With: Dr. Jerome Gross   Subject: Music

Lesson CLXX.

TWO-PART MELODIZATION

This technique consists of writing two correlated melodies (two-part counterpoint) to a given chord-progression. The counterpoint itself must satisfy all the requirements pertaining to harmonic intervals. Each of the melodic parts (to be designated as M_{I} and M_{II}, or as CP_{I} and CP_{II}) must satisfy the requirements pertaining to melodization.

The sequence in which two-part melodization should be performed is as follows:

(1) the writing of $H^{\rightarrow}$;

(2) the writing of M with the least number of attacks per H;

(3) the writing of M with the most number of attacks per H.

It is not essential which melody is designated as M_{I} and which as M_{II} .

Considering the natural physical scale of frequencies as increasing in the upward direction of musical pitch, we shall evolve the melody with the least number of attacks immediately above harmony, and the

melody with the most number of attacks above the first melody. Such schemes will be considered fundamental and could be later rearranged.

Thus we arrive at the two possible settings:

(1) $\frac{\frac{M_I}{M_{II}}}{H\rightarrow}$ and (2) $\frac{\frac{M_{II}}{M_I}}{H\rightarrow}$

Octave-convertibility (exchange of the positions of M_I and M_{II}) is possible only when the harmonic intervals of both melodic parts are chosen with consideration of such a convertibility. This mainly concerns the necessity of supporting certain higher functions (such as 11) by the immediately preceding function (such as 9).

All forms of quadrant rotation (ⓐ, ⓑ, ⓒ and ⓓ) are acceptable on one condition: M_I and M_{II} always remain above the chord progression ($H\rightarrow$).

As melodization of harmony by means of one part produced different types of melody in relation to the different types of harmonic progressions, the same possibilities still exist for the two-part melodization.

It is to be remembered that some types of melody in one-part melodization were the outcome of new techniques. For instance, the technique of modulating symmetric melody above all forms of symmetric harmony, or the technique of diatonic melody evolved from a quantitative scale above

all forms of chromatic harmony. All such new techniques shall be applied now to the two-part melodization. This, naturally, will result in the new types of counterpoint.

The distribution of attacks of M_I, M_{II} and $H\rightarrow$ is a matter of considerable complexity and will be discussed later. For the present, we shall distribute the attacks for all three parts (M_I, M_{II} and $H\rightarrow$) uniformly and by means of multiples.

Some elementary forms of the distribution of attacks.

M_I	a	2a	a	3a	a	4a	a	4a	2a	6a	2a	8a	2a	6a	3a	8a	4a
M_{II}	a	a	2a	a	3a	a	4a	2a	4a	2a	6a	2a	8a	3a	6a	4a	8a
$H\rightarrow$	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H

M_I	9a	3a	12a	3a	12a	4a	15a	3a	16a	4a	. . .
M_{II}	3a	9a	3a	12a	4a	12a	3a	15a	4a	16a	
$H\rightarrow$	H	H	H	H	H	H	H	H	H	H	

Here the quantities of attacks in $\frac{M_I}{M_{II}}$ are designated per chord.

Each original setting of two simultaneous melodies accompanied by a chord-progression offers seven forms of exposition.

(1) M_I; (2) M_{II}; (3) $H\rightarrow$; (4) $\frac{M_I}{M_{II}}$; (5) $\frac{M_I}{H\rightarrow}$; (6) $\frac{M_{II}}{H\rightarrow}$; (7) $\frac{M_I}{\frac{M_{II}}{H\rightarrow}}$

Melodization of Diatonic Harmony by means of Two-Part Diatonic Counterpoint.

(Type I and II)

The melody with least number of attacks and appearing immediately above harmony must conform with the principles of diatonic melodization. It is desirable not to include higher functions (9, 11) into this melody (we shall call it M_{II}), for the reason that the latter could be spared for the use in melody with the most number of attacks (we shall call it M_I). Thus the high functions of M_I will be supported by M_{II}. Scales of both melodies must have common source of derivation. This common source is the diatonic scale of harmony. Any derivative scales of the original d can be employed. Harmony can be devised in four or five parts. Four-part harmony is preferable as the texture of a duet accompanied by five parts is somewhat heavy.

None of the melodies must produce consecutive octaves with any of the harmonic parts.

M_I should be written as counterpoint to M_{II} and as melodization of the chord-progression.

Identical as well as non-identical scales (which derive through permutation of the pitch-units of d_o) can be used in M_I, M_{II} and $\overrightarrow{H}$. Under such conditions any d_o produces 35 possibilities of modal relations between the abovementioned three components.

As we are employing seven-unit scales,

$$_7C_3 = \frac{7!}{3!(7-3)!} = \frac{5040}{6\cdot 24} = \frac{5040}{144} = 35$$

The number of two-part melodizations which is possible to evolve to one chord-progression (written in one definite d) is:

$$_7C_2 = \frac{7!}{2!(7-2)!} = \frac{5040}{2\cdot 120} = \frac{5040}{240} = 21$$

Examples of Diatonic Two-Part Melodization

Figure I.

(please see pages 6 and 7)

Chromatization of the Diatonic

Two-Part Melodization.

In order to produce a greater contrast between M_I and M_{II} either one can be subjected to chromatic variation. If desirable, both melodies can be used in their chromatic version.

Chromatic variation is achieved by means of passing or auxiliary chromatic tones.

Example of Chromatic Variation.

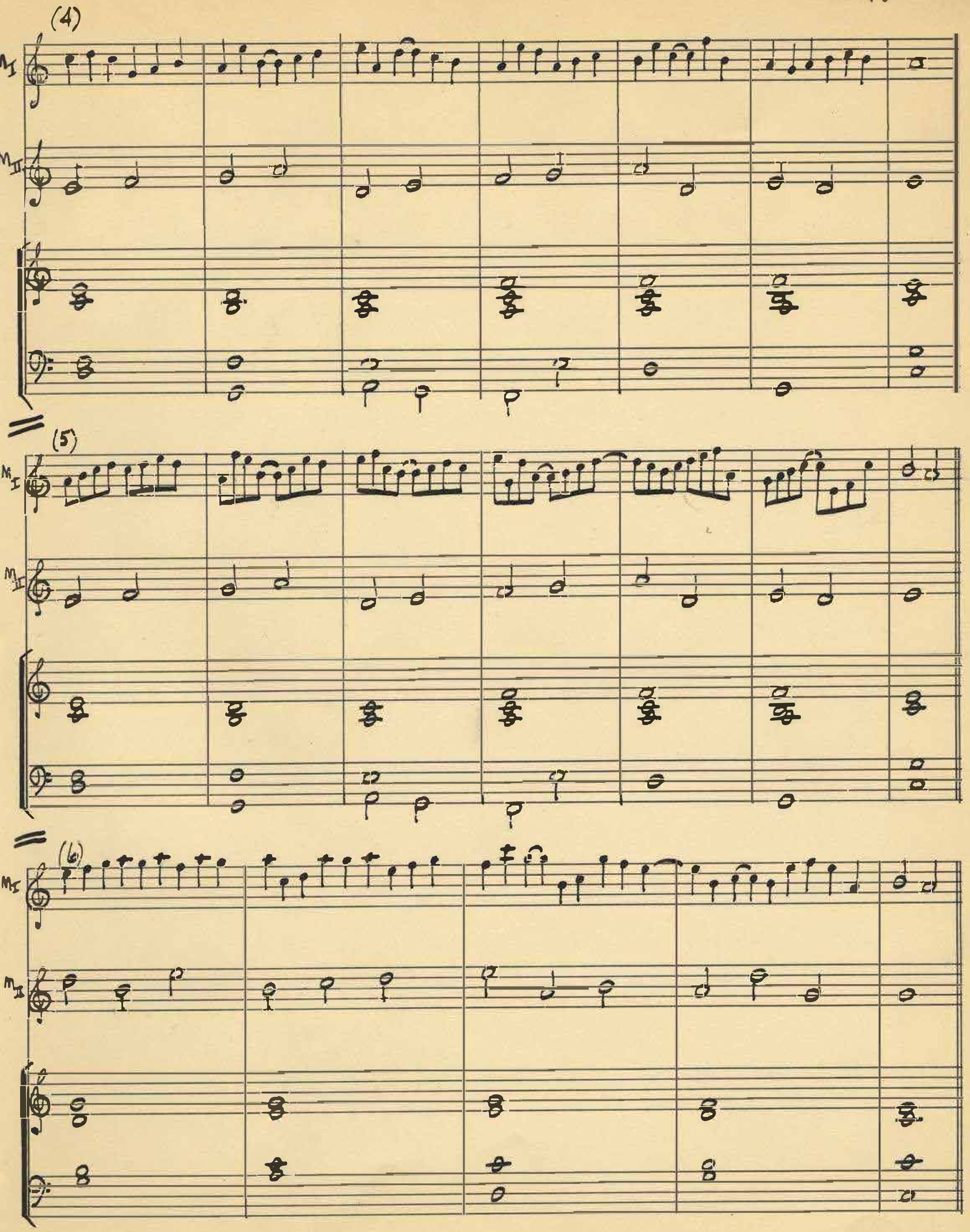
Figure II, Var. I and II.

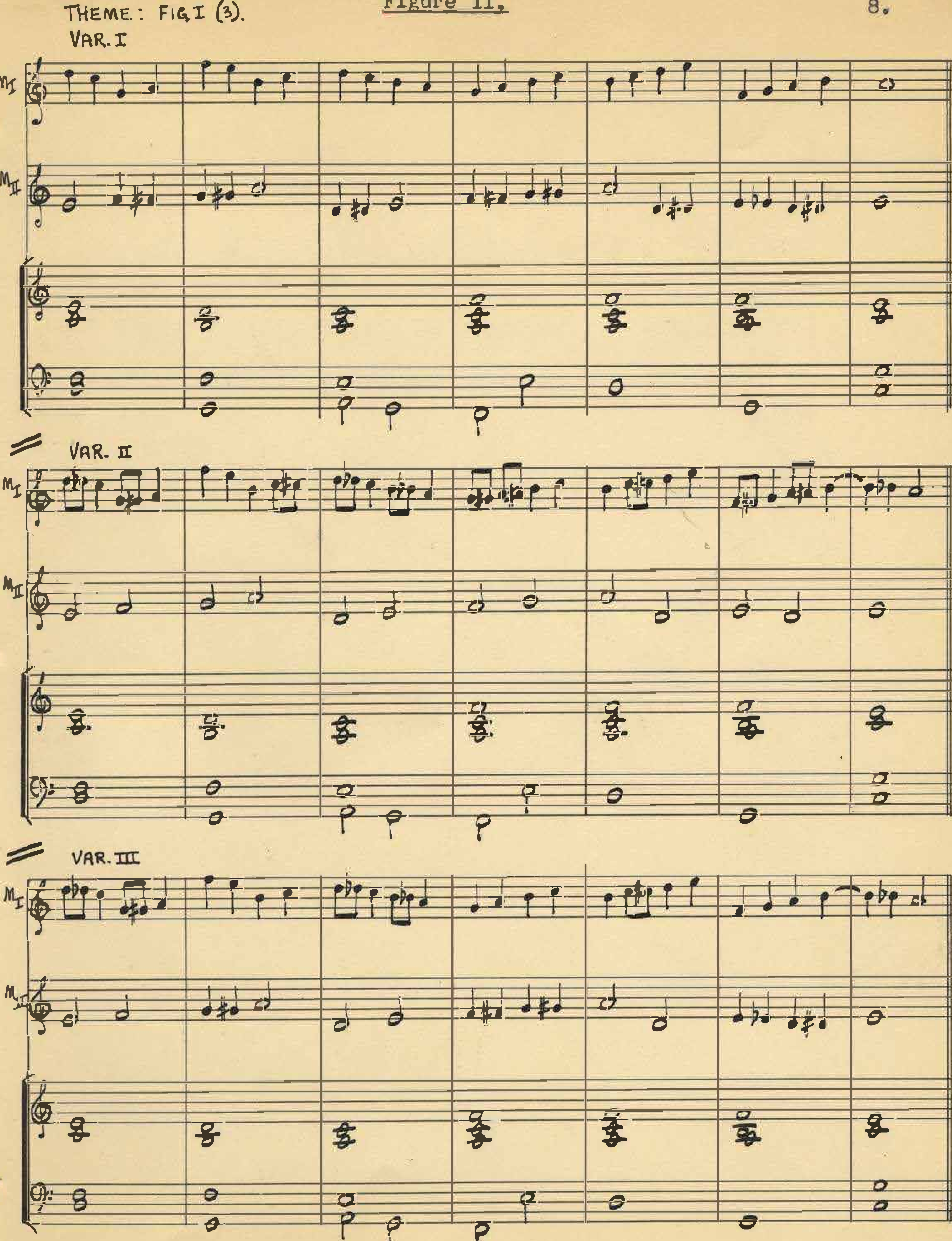
By means of combining the two variations of Fig. II, we can obtain a new version, where chromatic sections alternate with the diatonic ones.

Figure II, Var. III.

(please see page 8)

Figure I.


(4)
MI
MII
(5)
MI
MII
(6)
MI
MII


THEME: FIG I (3).
VAR. I
MI
MII
VAR. II
MI
MII
VAR. III
MI
MII

Lesson CLXXI.

Melodization of Symmetric Harmony (Type II, III and Generalized) by means of Two-Part Symmetric Counterpoint

Symmetric melodization is based on the pitch-scale which is the contracted Σ 13 corresponding to each individual H. Theoretically, each chord requires a new scale. The quality of the melody, however, depends on the quantity of common tones between the successive Σ 13 upon which the S are based. This concerns both M_I and M_{II} of the two-part melodization.

The ultimate requirements for two-part symmetric melodization may be stated as follows:

(1) Adherence of one M to a particular set of pitch-units thus producing a scale.

(2) The graduality of modulation, which is executed by means of common tones, chromatic alterations and identical motifs.

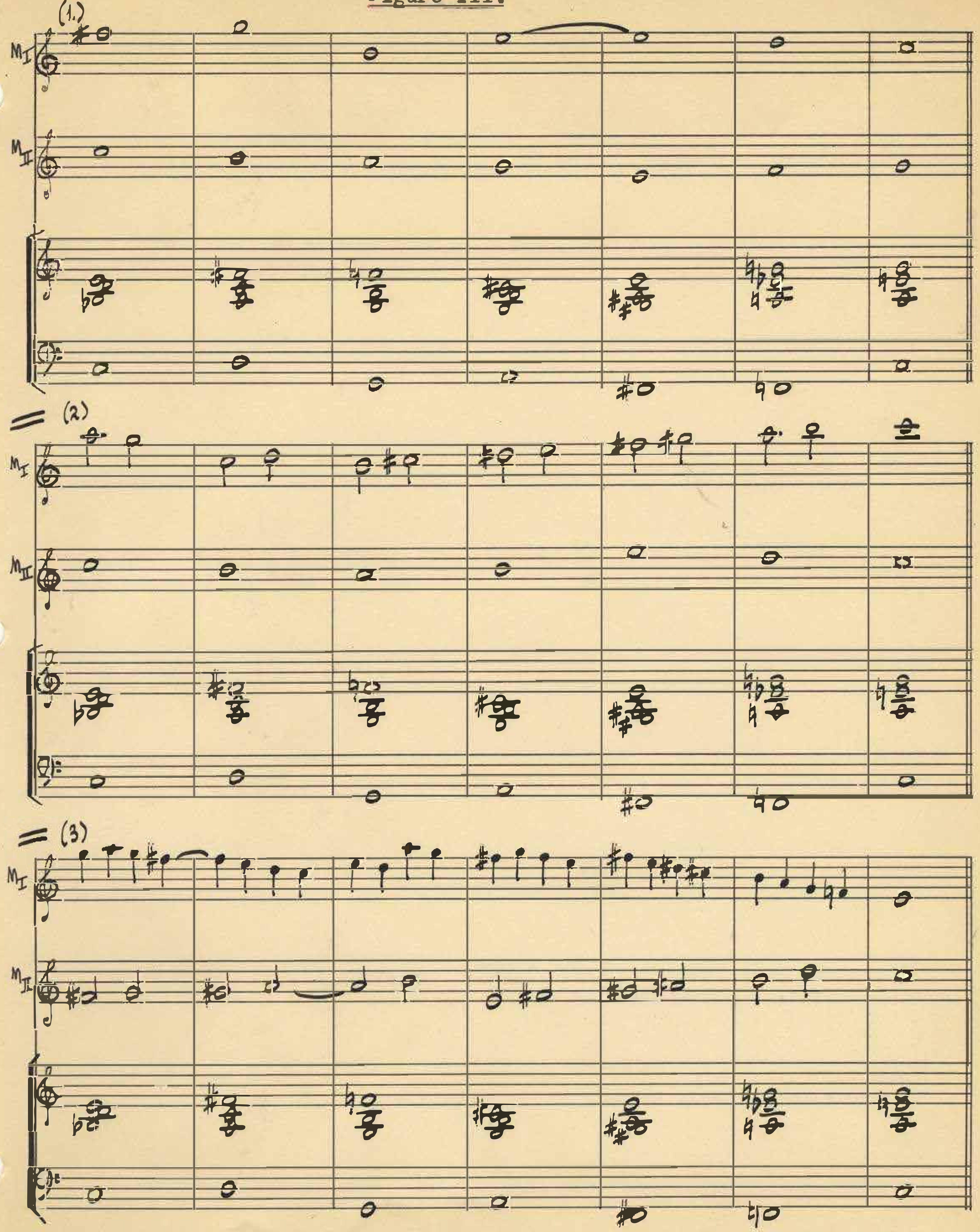
(3) Strict adherence to contrapuntal treatment of harmonic intervals between M_I and M_{II}.

Examples of Symmetric Two-Part Melodization

Figure III.

(please see next pages)
10 and 11

Figure III.




(4)
MI
MII
(5)
MI
MII
(6)
MI
MII

Chromatization of the Symmetric Two-Part Melodization.

This technique is identical with chromatization of the diatonic counterpoint. Passing and auxiliary chromatic tones are not the part of $\sum$ 13. Either of the two contrapuntal parts can be chromatized. Alternation of chromatic and symmetric sections in both melodies is fully satisfactory.

Example of Chromatic Variation

Figure IV.

(please see next page)


VAR. I
MI
MII
VAR. II
MI
MII
VAR. III
MI
MII

Lesson CLXXII.

Melodization of Chromatic Harmony by means of Two-Part Counterpoint

As one-part melodization of chromatic harmony is possible from two distinctly different sources:

(1) directional units and

(2) quantitative scale,

chromatic melodization in two parts is possible in the following combinations of the above techniques:

M_I	di	ch	di	ch
M_{II}	di	di	ch	ch
$H^{\rightarrow}$	ch	ch	ch	ch

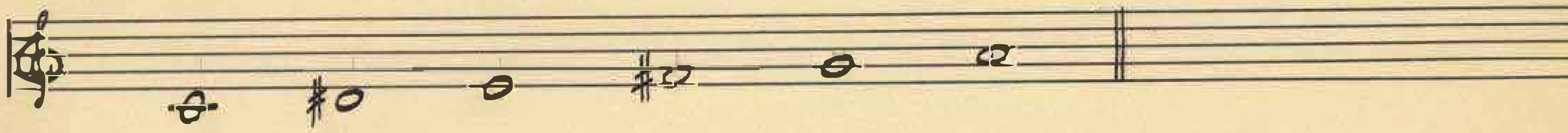
where di (diatonic) represents the quantitative scale; ch of M represents the directional units method and ch of $H^{\rightarrow}$ stands for chromatic harmonic continuity.

If there is a contrast to be achieved between M_I and M_{II}, one of them becomes di and the other ch.

If a similarity is preferable (the contrast still can be achieved by juxtaposition of the quantities of attacks of $\frac{M_I}{M_{II}}$) both melodies are either di or ch. The first has a diatonic character (due to adherence to one particular pitch-scale) and the second has a modulating character abundant with semitonal directional units.

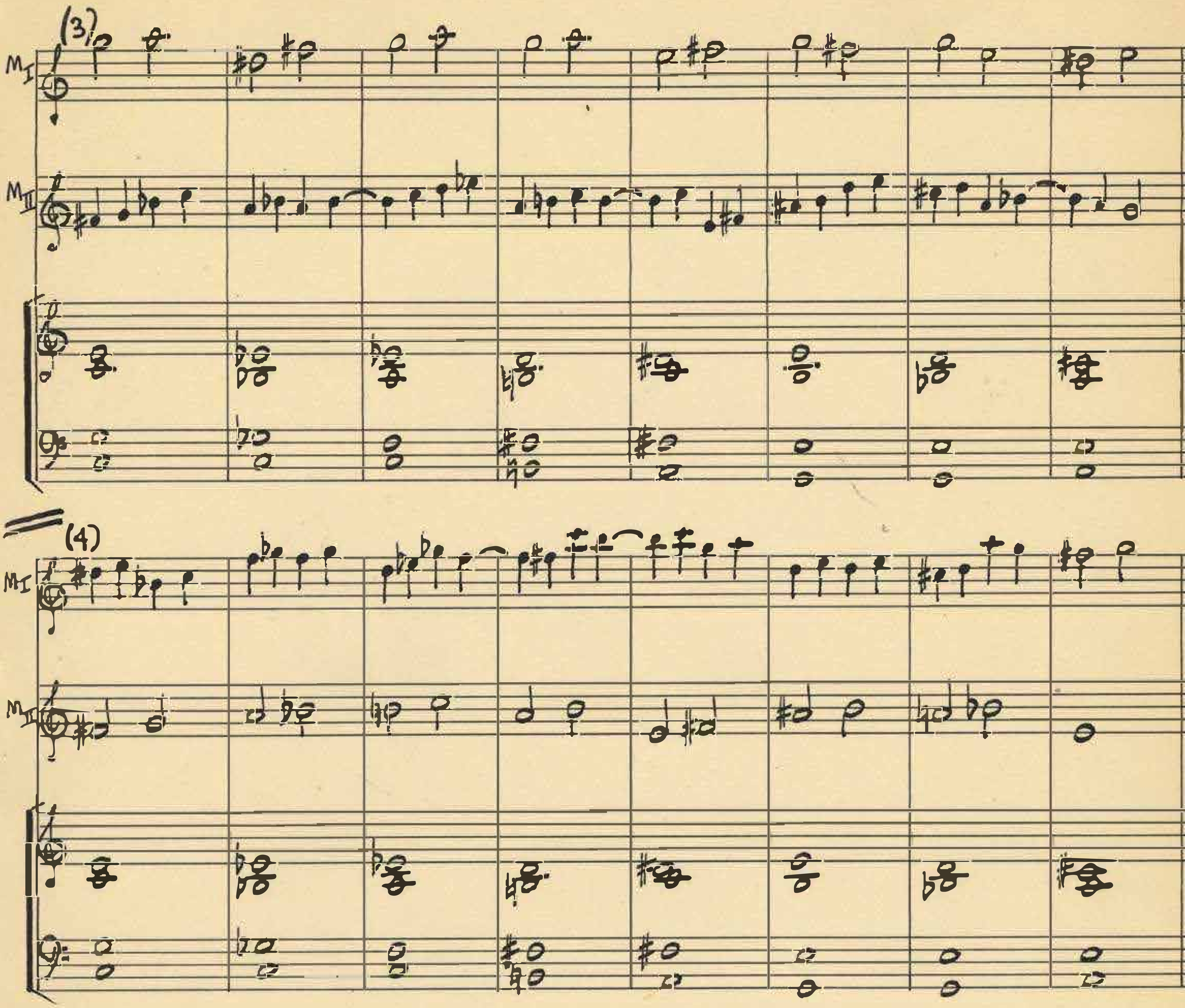
PROGRESSION: H→ch

QUANTITATIVE SCALE: S→ di



(1.)

(2.)


(3)
MI
MII
(4)
MI
MII

Lesson CLXXIII.

Composition of Attack-Groups for the Two-Part Melodization

The quantity of attacks of $\frac{\frac{M_I}{M_{II}}}{H}$ can be either constant or variable.

A constant form of the attack-group takes place when every individual H has a definite corresponding number of attacks in M_I and M_{II}, which remains the same for every consecutive H.

$$\frac{\frac{M_I}{M_{II}}}{H} = A \text{ const.}$$

A constant A does not necessitate an even distribution in $\frac{a(M_I)}{a(M_{II})}$. An even distribution may be considered merely as a special case.

Examples of an even distribution of A:

M_I	4a	6a	6a	8a	8a	9a	12a	12
M_{II}	2a	2a	3a	2a	4a	3a	3a	4a
H	a	a	a	a	a	a	a	a

Examples of uneven distribution of A:

M_I	2a+3a	4a+2a	4a+2a	4a+6a
M_{II}	a+a	a+a	2a+a	2a+2a
H	a	a	a	a

M_I	4a+2a+3a+6a	6a+3a+6a+4a+2a+9a
M_{II}	2a+a+a+2a	3a+a+2a+2a+a+3a
H	a	a

A variable form of the attack-group takes place when A emphasizes a group of chords, and when each consecutive H has a specified number of attacks for a definite quantity of chords.

For example: $\overrightarrow{A} = A_1 + A_2 + A_3$

Let $A_1 = \dfrac{M_I}{\dfrac{M_{II}}{H}} = \dfrac{2a+a}{\dfrac{a+a}{a}}$ and let $A_2 = \dfrac{M_I}{\dfrac{M_{II}}{H}} = \dfrac{4a+3a}{\dfrac{2a+a}{a}}$

and let $A_3 = \dfrac{M_I}{\dfrac{M_{II}}{H}} = \dfrac{4a+6a+3a}{\dfrac{2a+2a+a}{a}}$ then:

$$\frac{M_I}{\dfrac{M_{II}}{\overrightarrow{H}}} = \left(\frac{2a+a}{\dfrac{a+a}{a}}\right) H_1 + \left(\frac{4a+3a}{\dfrac{2a+a}{a}}\right) H_2 + \left(\frac{4a+6a+3a}{\dfrac{2a+2a+a}{a}}\right) H_3$$

All other considerations concerning the distribution and quantities of attacks are identical with one-part melodization (see: "Composition of the Attack-Groups of Melody" in the branch of Melodization of Harmony).

Example of Correlated Attack-Groups

in Two-Part Melodization

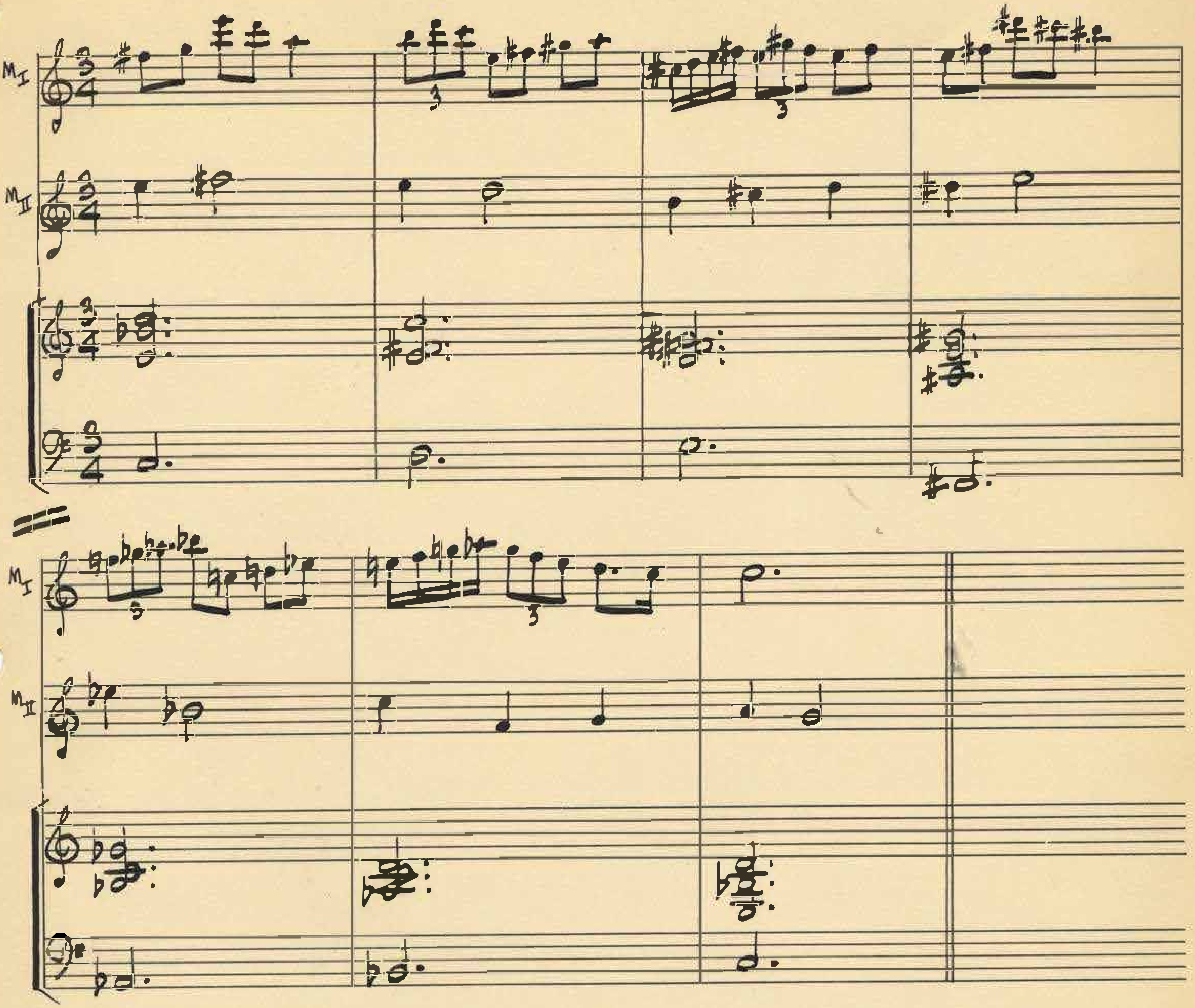
Figure VI.

$$\frac{M_I}{\dfrac{M_{II}}{\overrightarrow{H}}} = \left(\frac{2a+3a}{\dfrac{a+a}{a}}\right) H_1 + \left(\frac{3a+4a}{\dfrac{a+a}{a}}\right) H_2 + \left(\frac{4a+3a+2a}{\dfrac{a+a+a}{a}}\right) H_3$$

$\overrightarrow{H} = 6\sqrt[6]{2}$, S(9) const.; $\sum 13$ XIII; $S = \frac{3p}{p}$; transformation: $\leftrightarrows$

$T'' = 12t$ in $\frac{3}{4}$ time.

Figure VI.



Composition of Durations for the Attack-Groups of Two-Part Melodization

Selection of durations and duration-groups satisfying the attack-groups composed for two-part melodization can be based either on the Series of the Evolution of Rhythm Families (in which case there is no interference between the attacks of the attack-group and the attacks of the duration-group) or on a direct composition of duration-groups (which may or may not produce an interference between the attacks of the attack-group and the attack of the duration-group) which would be superimposed upon the attack-groups.

When the respective attack-groups are represented by the durations selected from Style-Series, and the number of individual attacks in the attack-sub-groups does not correspond to the number of attacks in the duration-groups, it is necessary to split the respective duration-units. This consideration concerns the first technique only (i.e., the matching of attack-groups by the series of durations).

Musical example of Figure VI is a translation of its corresponding attack-group into $\frac{3}{3}$ series, where three types of split-unit groups were used: $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. One exception to the series was made at the cadence, where a musical quarter was split into $\frac{4}{4}$ series binomial, i.e., 3+1.

The numerical representation of this example of melodization appears as follows:

$$\frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} = \left(\frac{\frac{1/2t+1/2t+1/2t+1/2t+t}{t \quad + 2\,t}}{3t}\right) H_1 +$$

$$+ \left(\frac{\frac{1/3t+1/3t+1/3t+1/2t+1/2t+1/2t+1/2t}{t \qquad + 2\,t}}{3t}\right) H_2 +$$

$$+ \left(\frac{\frac{1/4t+1/4t+1/4t+1/4t+1/3t+1/3t+1/3t+1/2t+1/2t}{t \qquad + \quad t \qquad + \quad t}}{3\,t}\right) H_3$$

The abundance of split units and split-unit groups in this instance is due to the abundance of attacks over each H and to relatively low value of the series. With a series of higher value, the splitting of units would be greatly reduced.

We shall translate now the same example into $\frac{9}{9}$ series:

$$\frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} = \left(\frac{\frac{t+3t+t+3t+t}{4t \quad +5t}}{9t}\right) H_1 + \left(\frac{\frac{t+2t+t+t+2t+t+t}{4t \quad +5t}}{9t}\right) H_2 +$$

$$+ \left(\frac{\frac{t+t+t+t+t+t+t+t+t}{4t \quad +3t \quad +2t}}{9t}\right) H_3$$

Figure VII.

(please see next page)

Figure VII.

Now we shall take a case where the attack and the duration-groups are composed independently.

Let $r_{5\div4}$ represent the quantities of attacks of M_I to each attack of M_{II}, and let every 2 attacks of M_{II} correspond to one attack of $\overrightarrow{H}$.

Then the distribution of attacks for all three parts takes the following appearance:

$$\frac{a(M_I)}{\frac{a(M_{II})}{a(\overrightarrow{H})}} = \left(\frac{4a+a}{\frac{a+a}{a}}\right)H_1 + \left(\frac{3a+2a}{\frac{a+\ a}{a}}\right)H_2 + \left(\frac{2a+3a}{\frac{a+\ a}{a}}\right)H_3 + \left(\frac{a+4a}{\frac{a+\ a}{a}}\right)H_4$$

Let us superimpose the following duration-group:

$$T = \underline{r_{4\div3}} = 16t;\ 10a$$

Then: $\frac{a(A)}{a(T)} = \frac{20}{10} = \frac{2}{1}$; $\begin{matrix}1(20)\\2(10)\end{matrix}$

Hence, $T' = 16t \cdot 2 = 32t$

Let $T'' = 8t$, then: $N_{T''} = \frac{32}{8} = 4$

Each $a(M_I)$ corresponds to an individual term of T; each $a(M_{II})$ corresponds to the sum of the respective durations of M_I; each $a(\overrightarrow{H})$ corresponds to the sum of 2 durations of M_{II}.

The final temporal scheme of this two-part melodization takes the following form:

$$\frac{M_I}{\frac{M_{II}}{\overrightarrow{H}}} = \left(\frac{3t+t+2t+t+t}{\frac{7t\qquad +t}{8t}}\right)H_1 + \left(\frac{t+t+2t+t+3t}{\frac{4t\qquad +4t}{8t}}\right)H_2 +$$

$$+ \left(\frac{3t+t+2t+t+t}{\frac{4t\quad +4t}{8t}}\right)H_3 + \left(\frac{t+t+2t+t+3t}{\frac{t+7t}{8t}}\right)H_4$$

Figure VIII.

Direct Composition of Durations for the Two-Part Melodization

Direct composition of durations becomes particularly valuable, when a proportionate distribution of durations for a constant number of attacks between the component parts (M_I, M_{II} and $H\rightarrow$) is desired. Distributive involution of three synchronized powers solves this problem. As it follows from the Theory of Rhythm, the cube of a binomial produces an eight-term polynomial, the square of a binomial produces a quadrinomial and the first-power group remains a binomial. Thus, the quantity of attacks of the two adjacent parts $\frac{M_I}{M_{II}}$ and $\frac{M_{II}}{H\rightarrow}$ is two. Cubing of a trinomial gives a twenty-seven-term polynomial, the synchronized square producing nine and the first-power group -- three terms. The quantity of attacks between the two adjacent parts remains three. Thus, the number of terms of the original polynomial equals the quantity of attacks between the adjacent parts.

We shall devise now a correlated proportionate system of duration-groups. The distributive cube will serve as T for M_I, the synchronized distributive square as T for M_{II} and the synchronized first-power group as T for $H\rightarrow$.

We shall operate from the trinomial of the $\frac{4}{4}$ series. This secures the following attack-group correlation:

$$\frac{\dfrac{a(M_I)}{a(M_{II})}}{a(\overrightarrow{H})} = \frac{\dfrac{9a}{3a}}{a}$$

The entire temporal scheme assumes the following form:

$$\frac{\dfrac{T(M_I)}{T(M_{II})}}{T(\overrightarrow{H})} = \frac{\dfrac{[(8t+4t+4t) + (4t+2t+2t) + (4t+2t+2t)]}{(16t + 8t + 8t)}}{32tH_1} + $$

$$+ \frac{\dfrac{[(4t+2t+2t) + (2t+t+t) + (2t+t+t)]}{(8t + 4t + 4t)}}{16tH_2} + $$

$$+ \frac{\dfrac{[(4t+2t+2t) + (2t+t+t) + (2t+t+t)]}{(8t + 4t + 4t)}}{16tH_3}$$

<u>Figure IX.</u>

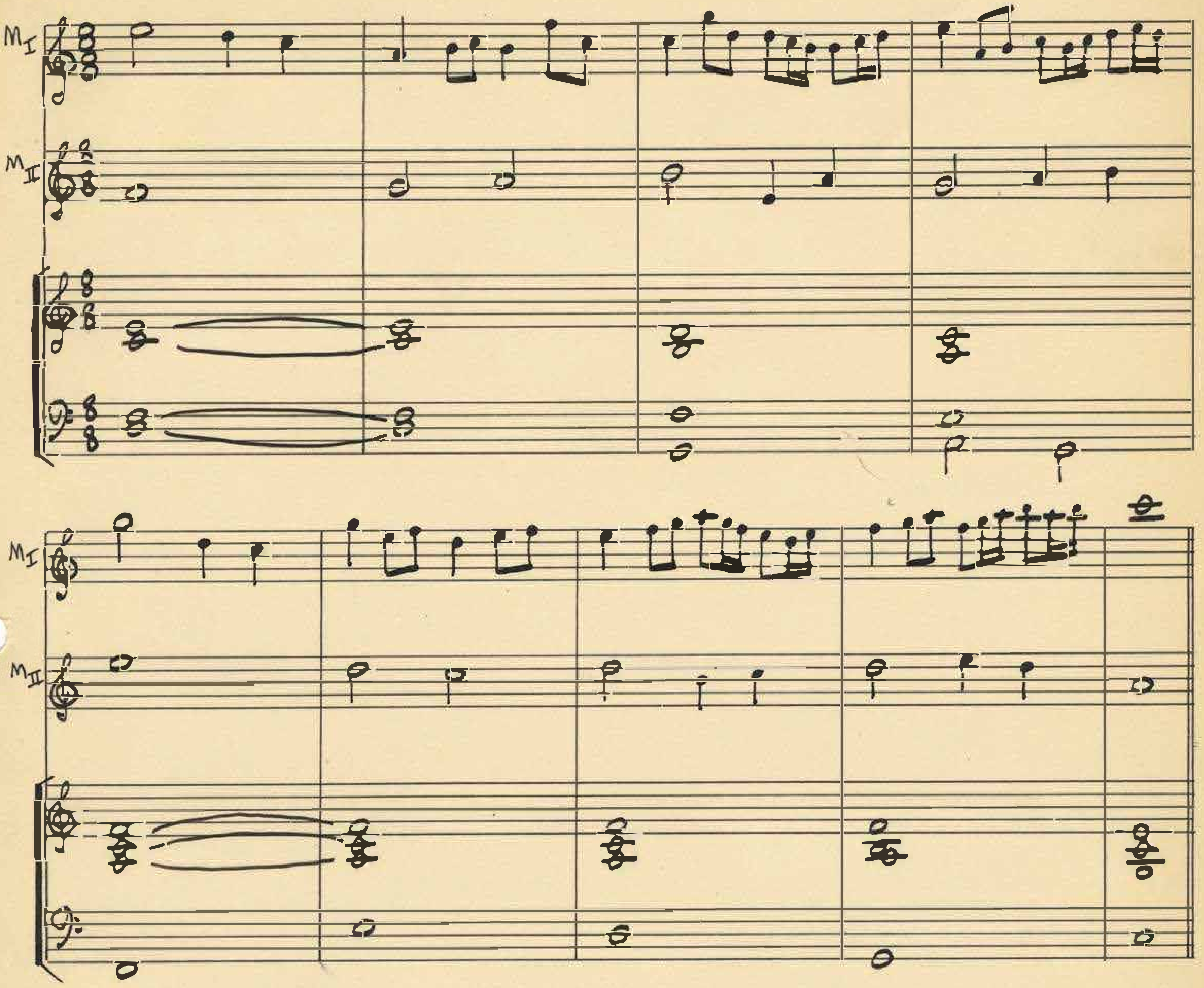
(please see page 27)

In addition to this technique, <u>coefficients of duration</u> can be used for correlation of durations in the two-part melodization.

Example:

$$\frac{\dfrac{M_I}{M_{II}}}{\overrightarrow{H}} = \frac{\dfrac{(3t+t+2t+2t)+(3t+t+2t+2t)+(3t+t+2t+2t)+(3t+t+2t+2t)}{(6t+2t+4t+4t) + (6t+2t+4t+4t)}}{12tH_1 + 4tH_2 + 8tH_3 + 8tH_4}$$

Figure IX.



Lesson CLXXIV.

Composition of Continuity in Two-Part Melodization

The seven forms of expositions previously classified can be now incorporated into continuity of two-part melodization. The applied meaning of these seven forms can be expressed as follows:

(1) M_I -- Solo melody: theme A;

(2) M_{II} -- Solo melody: theme B;

(3) $H^{\rightarrow}$ -- Solo harmony: theme C;

(4) $\frac{M_I}{H^{\rightarrow}}$ -- Solo melody with harmonic accompaniment (theme A accompanied);

(5) $\frac{M_{II}}{H^{\rightarrow}}$ -- Solo melody with harmonic accompaniment (theme B accompanied);

(6) $\frac{M_I}{M_{II}}$ -- Duet of two melodies $\left(\frac{\text{Theme A}}{\text{Theme B}}\right)$

(7) $\frac{M_I}{\frac{M_{II}}{H^{\rightarrow}}}$ -- Duet of two melodies with harmonic accompaniment $\left(\frac{\text{Theme A}}{\frac{\text{Theme B}}{\text{Theme C}}}\right)$

The above seven forms serve as thematic elements of a composition, in which they appear in an organized sequence producing a complete musical whole.

Themes A, B and C must be considered as component parts of the whole in which they express their

particular characteristics. These characteristics which distinguish A from B and C are:

(1) High mobility of A (maximum quantity of attacks);

(2) Medium mobility of B (medium quantity of attacks);

(3) Low mobility of C (minimum quantity of attacks) combined with maximum density (four or five parts).

The planning of continuity must be based on a definite pattern of the variation of density combined with the variation of the quantity of attacks.

The scale of density can be arranged from low to high as follows:

(1) $A, \ \frac{A}{B}, \ C, \ \frac{A}{C}, \ \begin{array}{c}A\\\hline B\\\hline C\end{array};$

(2) $B, \ \frac{A}{B}, \ C, \ \frac{B}{C}, \ \begin{array}{c}A\\\hline B\\\hline C\end{array}.$

More or less extreme points of any such scale produce contrasts. For instance:

(1) $\begin{array}{c}A\\\hline B\\\hline C\end{array} + B + \begin{array}{c}A\\\hline B\\\hline C\end{array} + A + \begin{array}{c}A\\\hline B\\\hline C\end{array} + A + C + B + C + \begin{array}{c}A\\\hline B\\\hline C\end{array};$

(2) $A + C + B + C + A + \begin{array}{c}A\\\hline B\\\hline C\end{array} + B + \begin{array}{c}A\\\hline B\\\hline C\end{array} + A + B$

Durations corresponding to one individual attack of the component of lowest mobility (mostly $H^{\rightarrow}$) become time-units of the continuity. Such units (we shall call them T) can be arranged in any form of rhythmic distribution.

Correlation of the thematic duration-groups

(T's with their coefficients) with the different forms of density constitutes a composition.

Assuming that there are three forms of density and three forms of mobility, we obtain the following combined thematic forms (Low, Medium, High): $3^2 = 9$.

Density / Mobility	Low / Low	Low / Medium	Medium / Low	Medium / Medium	Low / High	High / Low	Medium / High	High / Medium	High / High

Thus, for instance: $\frac{\text{Density}}{\text{Mobility}} = \frac{\text{Low}}{\text{Low}} \equiv M_{II}$;

$\frac{\text{Density}}{\text{Mobility}} = \frac{\text{High}}{\text{Low}} \equiv H\!\rightarrow$; $\frac{\text{Density}}{\text{Mobility}} = \frac{\text{High}}{\text{Medium}} \equiv \frac{M_{II}}{H\rightarrow}$ etc.

We shall now devise a composition which will combine the gradual and the sudden variations of mobility and of density.

It is desirable to have such a scheme of two-part melodization which is cyclic and recapitulating, i.e., one permitting a correct transition from the end to the beginning for all three components.

For the present, we shall not resort to any additional techniques (such as inversions, expansions etc.), as the complete synthesis will be accomplished in the branch of Composition.

Let Figure VIII serve as the fundamental scheme of two-part melodization, as this material is

cyclic and recapitulating.

Let us adopt the following scheme of density and mobility:

$$\frac{\text{Density}}{\text{Mobility}} = \frac{\text{Low}}{\text{Low}} + \frac{\text{Low}}{\text{High}} + \frac{\text{Medium}}{\text{High}} + \frac{\text{High}}{\text{Medium}} + \frac{\text{High}}{\text{Low}} + \frac{\text{Medium}}{\text{High}} + \frac{\text{High}}{\text{High}}$$

The sequence of thematic elements and their combinations, corresponding to the seven forms of expositions and satisfying the above scheme of thematic forms may be selected as follows:

$$E^{\rightarrow} = M_{II}E_1 + M_IE_2 + \frac{M_I}{H^{\rightarrow}}E_3 + \frac{M_{II}}{H^{\rightarrow}}E_4 + H^{\rightarrow}E_5 + \frac{M_I}{M_{II}}E_6 + \left(\frac{M_I}{\frac{M_{II}}{H^{\rightarrow}}}\right)E_7.$$

We shall make T correspond to H and establish the following sequence for the T's: $T = r_{5\div3}$.

$$T^{\rightarrow} = T_1 3H + T_2 2H + T_3 H + T_4 3H + T_5 H + T_6 2H + T_7 3H$$

$$T^{\rightarrow} = 7T\ 15H.$$

The 7T of $T^{\rightarrow}$ produce no interference in relation to the 7E of $E^{\rightarrow}$. There is an interference between $T^{\rightarrow}E^{\rightarrow}$ and $H^{\rightarrow}$, however, as $H^{\rightarrow} = 8H$.

$$\frac{T^{\rightarrow}E^{\rightarrow}}{H^{\rightarrow}} = \frac{7}{8}; \quad \begin{matrix}8\\7\end{matrix}\begin{Bmatrix}7\\8\end{Bmatrix}; \quad E^{\rightarrow\prime} = 7\cdot8 = 56\ TE.$$

As 7 TE corresponds to 15 H, there will be 7 TE·8 = 56 TE and 15 H·8 = 120 H.

Thus the complete composition after synchronization evolves into the following form:

$$T^{\rightarrow\prime}E^{\rightarrow\prime} = 56\ TE\ 120\ H; \quad T'' = H\ ;\ N_{T''} = 120\ .$$

As in Figure VIII $T'' = TH$, the entire composition consumes 120 measures, which is 15 times the duration of the original scheme of melodization.

Here is the final layout of the composition:

Figure X.

$$T^{\rightarrow}E^{\rightarrow} = [M_{II}\,(H_1 + H_2 + H_3)\,T_1E_1 + M_I(H_4 + H_5)\,T_2E_2 + \frac{M_I}{H^{\rightarrow}}(H_6)T_3E_3 +$$

$$+ \frac{M_{II}}{H^{\rightarrow}}\,(H_7 + H_8 + H_1)\,T_4E_4 + H^{\rightarrow}(H_2)\,T_5E_5 + \frac{M_I}{M_{II}}\,(H_3 + H_4)T_6E_6 +$$

$$+ \frac{\frac{M_I}{M_{II}}}{H^{\rightarrow}}\,(H_5 + H_6 + H_7)\,T_7E_7] + [\,M_{II}\,(H_8 + H_1 + H_2)T_8E_8 +$$

$$+ M_I\,(H_3 + H_4)\,T_9E_9 + \frac{M_I}{H^{\rightarrow}}\,(H_5)\,T_{10}E_{10} + \frac{M_{II}}{H^{\rightarrow}}\,(H_6 + H_7 + H_8)$$

$$T_{11}E_{11} + H^{\rightarrow}(H_1)\,T_{12}E_{12} + \frac{M_I}{M_{II}}\,(H_2 + H_3)\,T_{13}E_{13} +$$

$$+ \frac{\frac{M_I}{M_{II}}}{H^{\rightarrow}}\,(H_4 + H_5 + H_6)\,T_{14}E_{14}\,] + [M_{II}\,(H_7 + H_8 + H_1)\,T_{15}E_{15} +$$

$$+ M_I\,(H_2 + H_3)\,T_{16}E_{16} + \frac{M_I}{H^{\rightarrow}}\,(H_4)\,T_{17}E_{17} +$$

$$+ \frac{M_{II}}{H^{\rightarrow}}\,(H_5 + H_6 + H_7)\,T_{18}E_{18} + H^{\rightarrow}(H_8)\,T_{19}E_{19} + \frac{M_I}{M_{II}}\,(H_1 + H_2)T_{20}E_{20} +$$

$$+ \frac{\frac{M_I}{M_{II}}}{H^{\rightarrow}}\,(H_3 + H_4 + H_5)\,T_{21}E_{21}\,] + [M_{II}\,(H_6 + H_7 + H_8)\,T_{22}E_{22} +$$

$$+ M_I\,(H_1 + H_2)\,T_{23}E_{23} + \frac{M_I}{H^{\rightarrow}}\,(H_3)\,T_{24}E_{24} +$$

$$+ \frac{M_{II}}{H^{\rightarrow}}\,(H_4 + H_5 + H_6)\,T_{25}E_{25} + H^{\rightarrow}(H_7)\,T_{26}E_{26} +$$

$$+ \frac{M_I}{M_{II}} (H_8 + H_1) T_{27} E_{27} + \frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} (H_2 + H_3 + H_4) T_{28} E_{28}] +$$

$$+ [M_{II} (H_5 + H_6 + H_7) T_{29} E_{29} + M_I (H_8 + H_1) T_{30} E_{30} +$$

$$+ \frac{M_I}{\overrightarrow{H}} (H_2) T_{31} E_{31} + \frac{M_{II}}{\overrightarrow{H}} (H_3 + H_4 + H_5) T_{32} E_{32} + \overrightarrow{H}(H_6) T_{33} E_{33} +$$

$$+ \frac{M_I}{M_{II}} (H_7 + H_8) T_{34} E_{34} + \frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} (H_1 + H_2 + H_3) T_{35} E_{35}] +$$

$$+ [M_{II} (H_4 + H_5 + H_6) T_{36} E_{36} + M_I (H_7 + H_8) T_{37} E_{37} +$$

$$+ \frac{M_I}{\overrightarrow{H}} (H_1) T_{38} E_{38} + \frac{M_{II}}{\overrightarrow{H}} (H_2 + H_3 + H_4) T_{39} E_{39} + \overrightarrow{H}(H_5) T_{40} E_{40} +$$

$$+ \frac{M_I}{M_{II}} (H_6 + H_7) T_{41} E_{41} + \frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} (H_8 + H_1 + H_2) T_{42} E_{42}] +$$

$$+ [M_{II} (H_3 + H_4 + H_5) T_{43} E_{43} + M_I (H_6 + H_7) T_{44} E_{44} +$$

$$+ \frac{M_I}{\overrightarrow{H}} (H_8) T_{45} E_{45} + \frac{M_{II}}{\overrightarrow{H}} (H_1 + H_2 + H_3) T_{46} E_{46} +$$

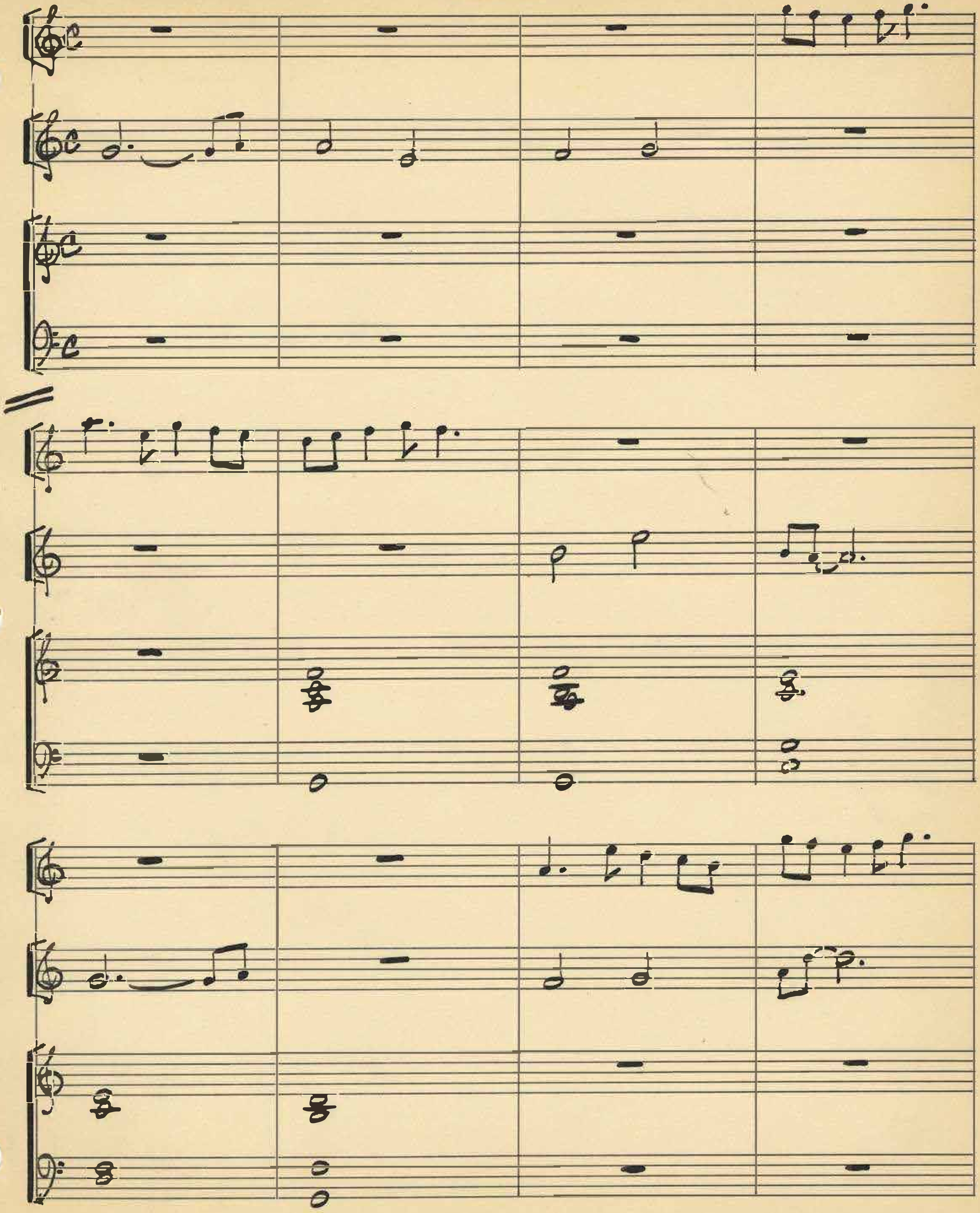
$$+ \overrightarrow{H}(H_4) T_{47} E_{47} + \frac{M_I}{M_{II}} (H_5 + H_6) T_{48} E_{48} + \frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} (H_7 + H_8 + H_1) T_{49} E_{49}] +$$

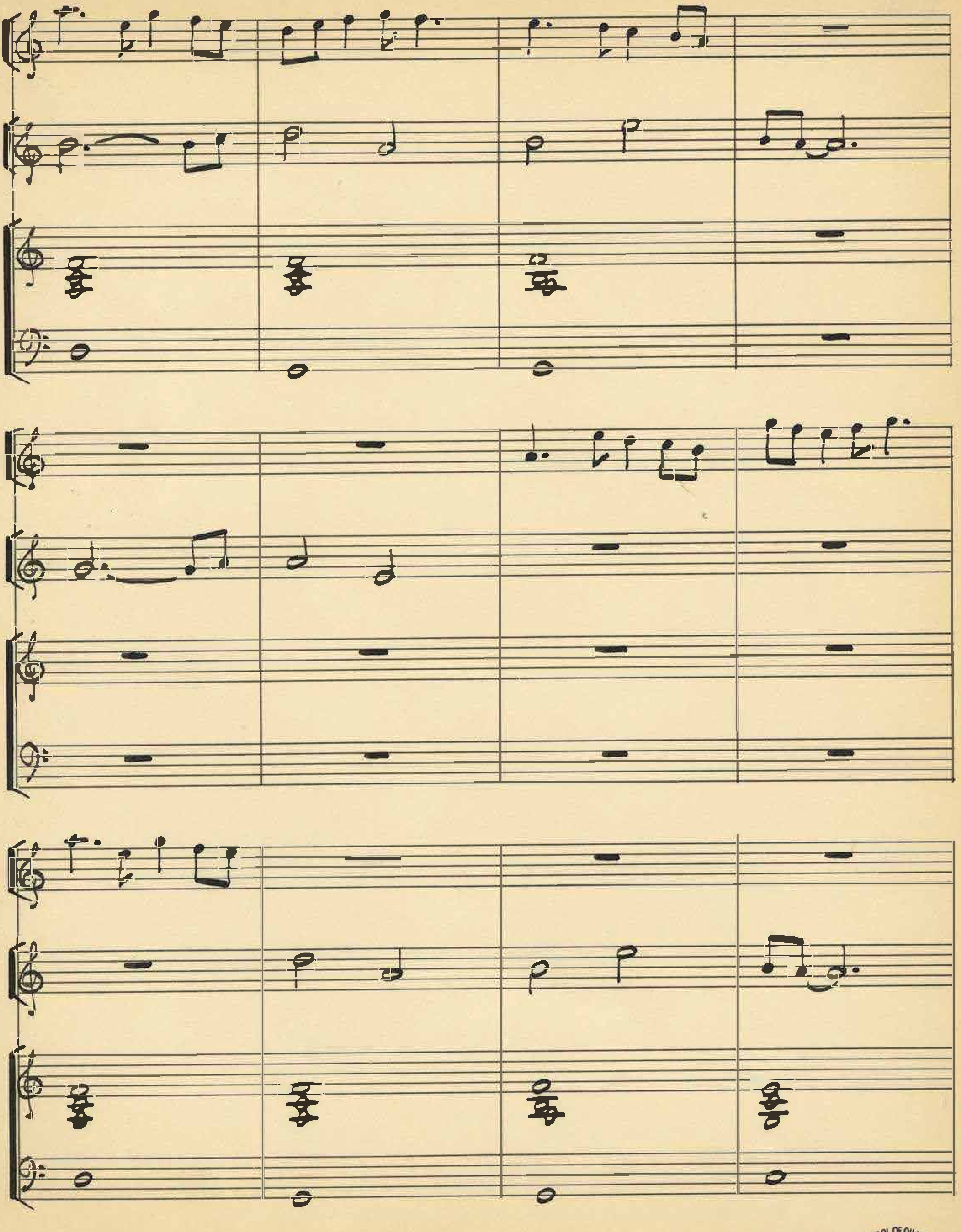
$$+ [M_{II} (H_2 + H_3 + H_4) T_{50} E_{50} + M_I (H_5 + H_6) T_{51} E_{51} +$$

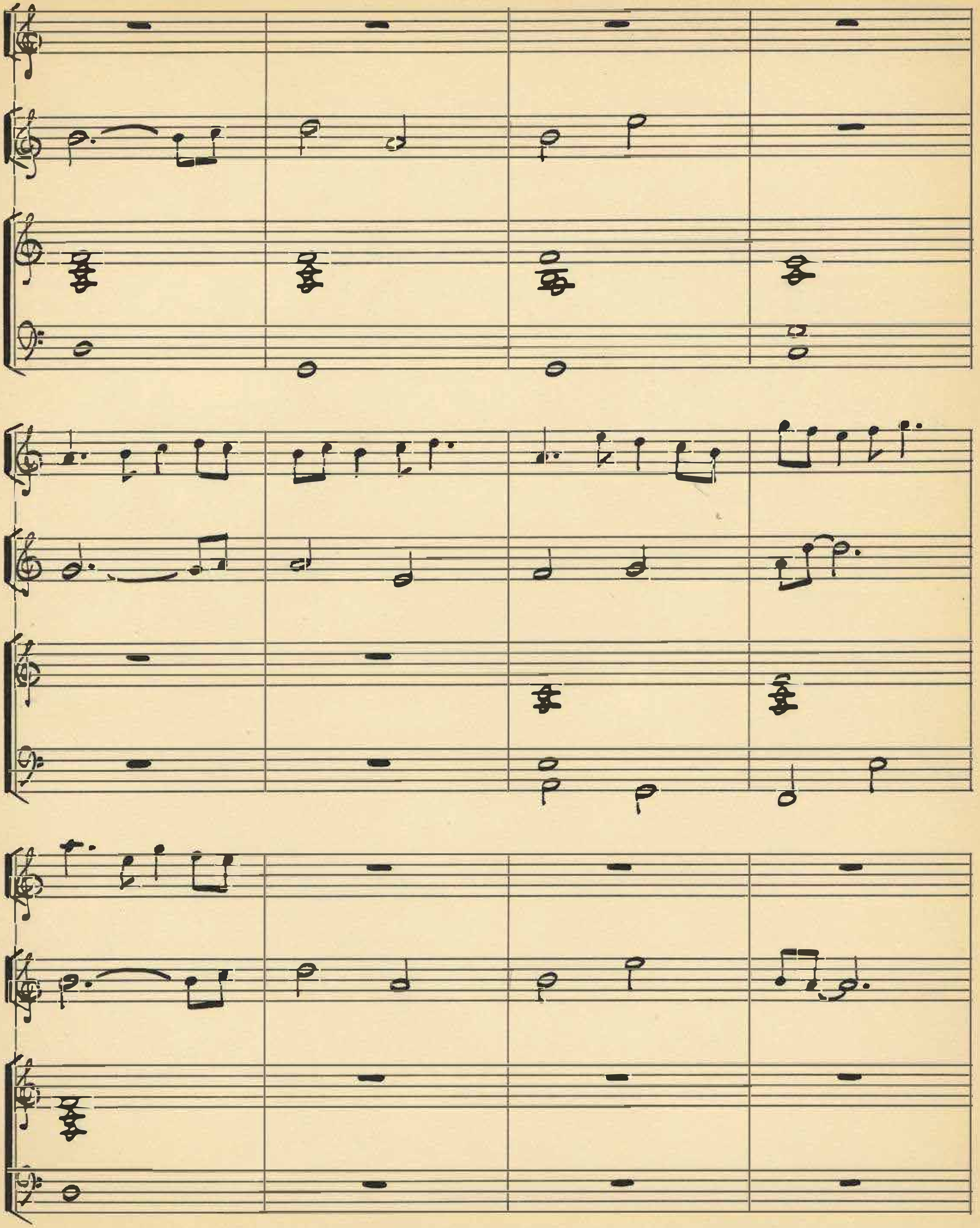
$$+ \frac{M_I}{\overrightarrow{H}} (H_7) T_{52} E_{52} + \frac{M_{II}}{\overrightarrow{H}} (H_8 + H_1 + H_2) T_{53} E_{53} + \overrightarrow{H}(H_3) T_{54} E_{54} +$$

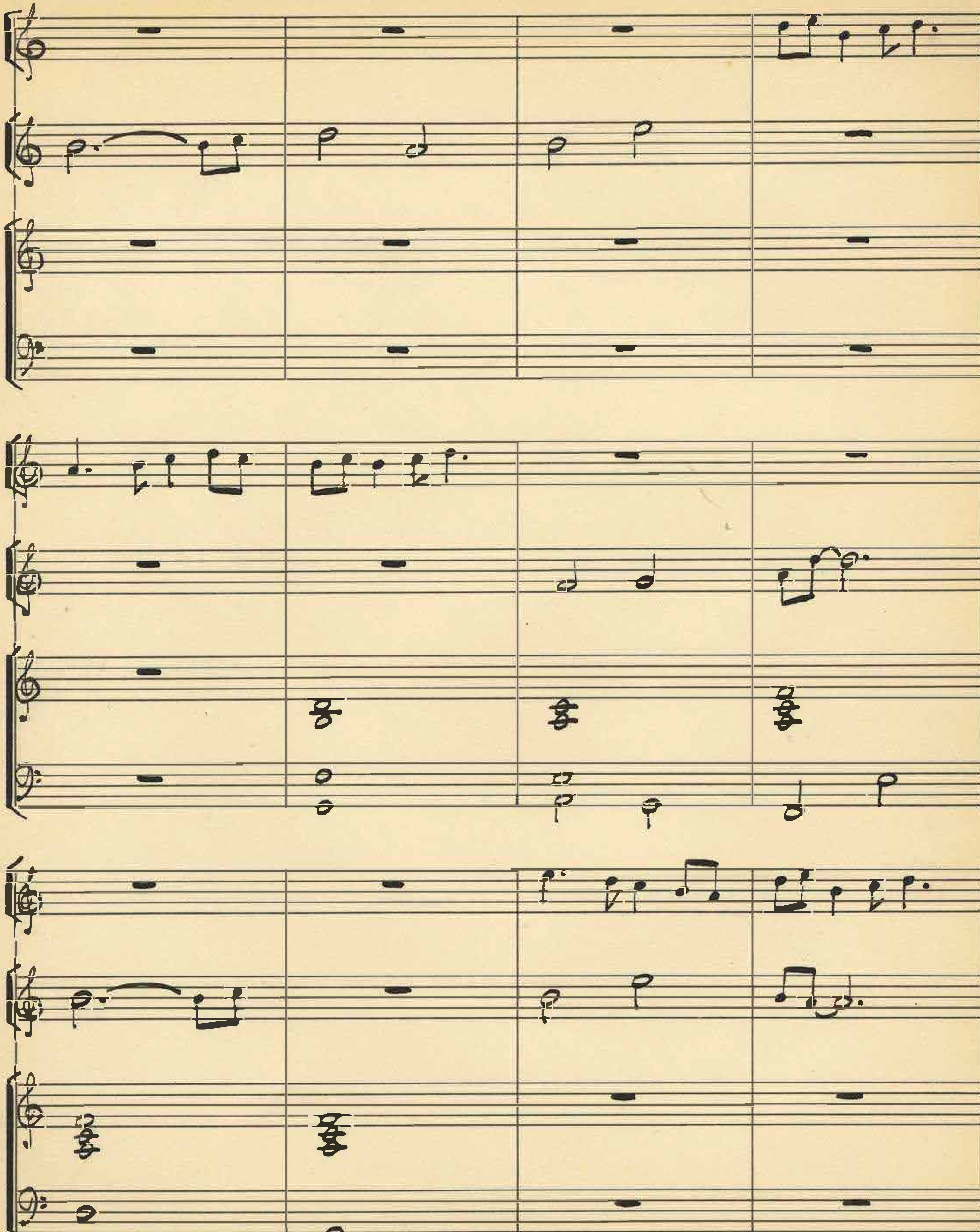
$$+ \frac{M_I}{M_{II}} (H_4 + H_5) T_{55} E_{55} + \frac{\frac{M_I}{M_{II}}}{\overrightarrow{H}} (H_6 + H_7 + H_8) T_{56} E_{56}].$$

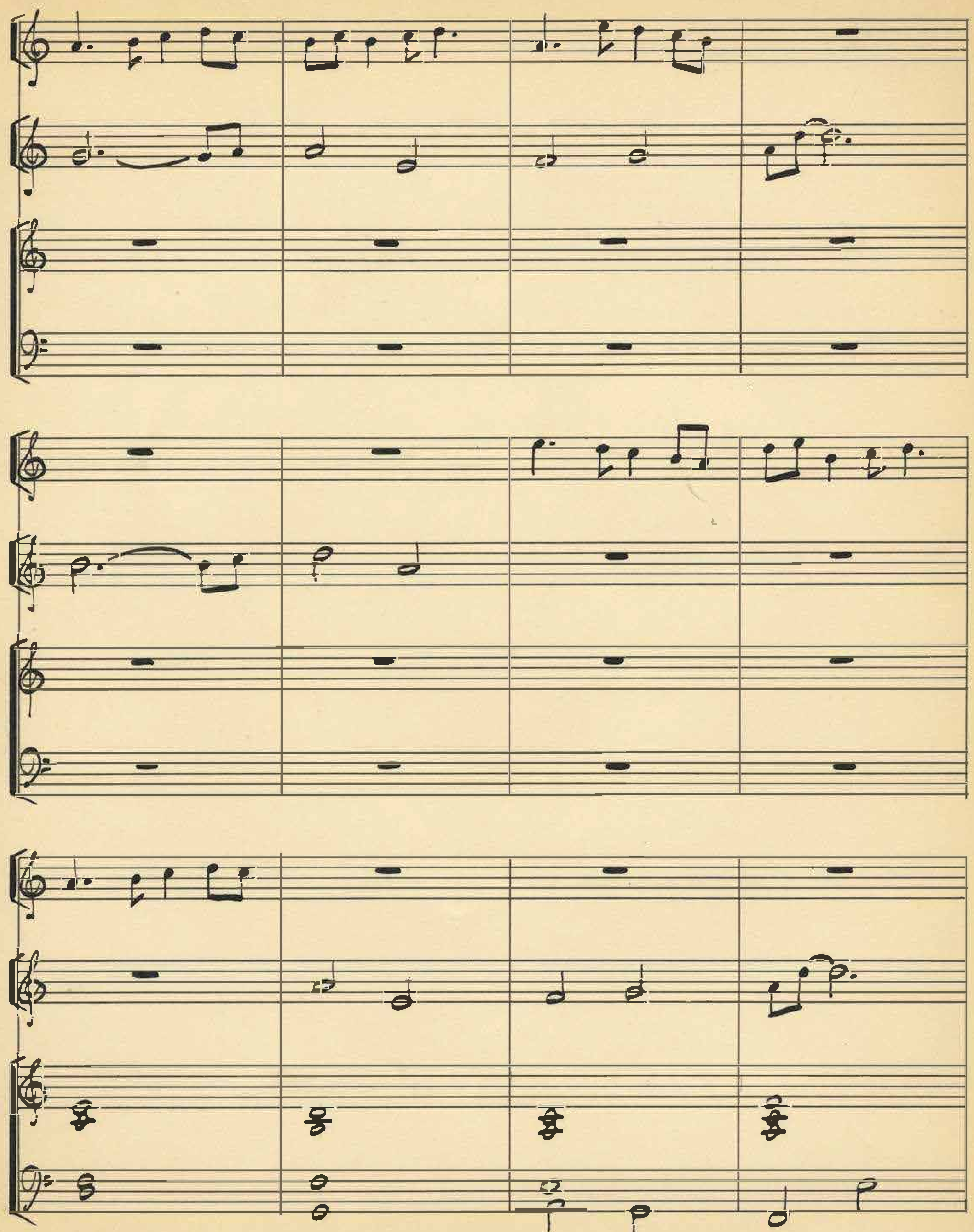
Lesson CLXXV. Figure X.

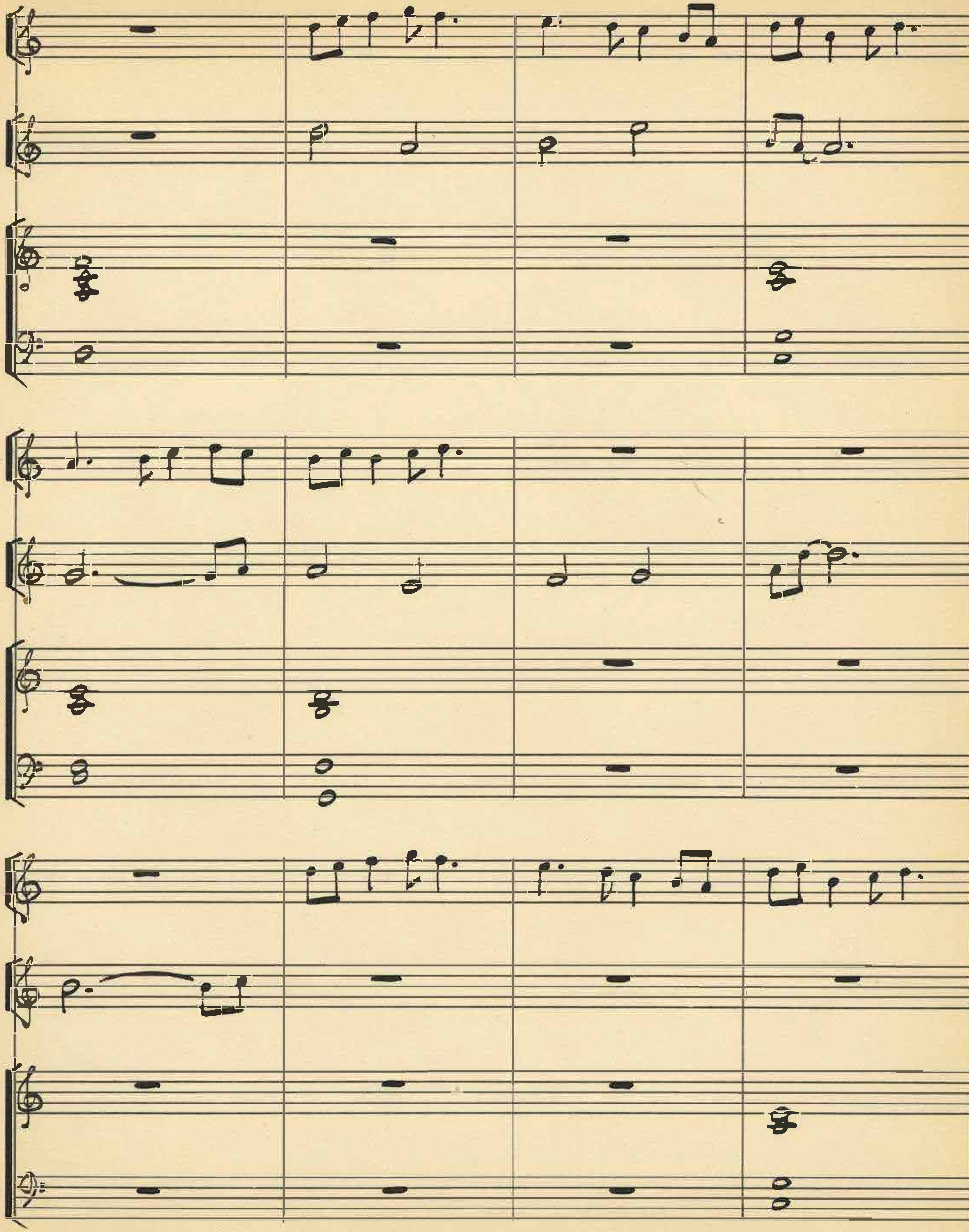


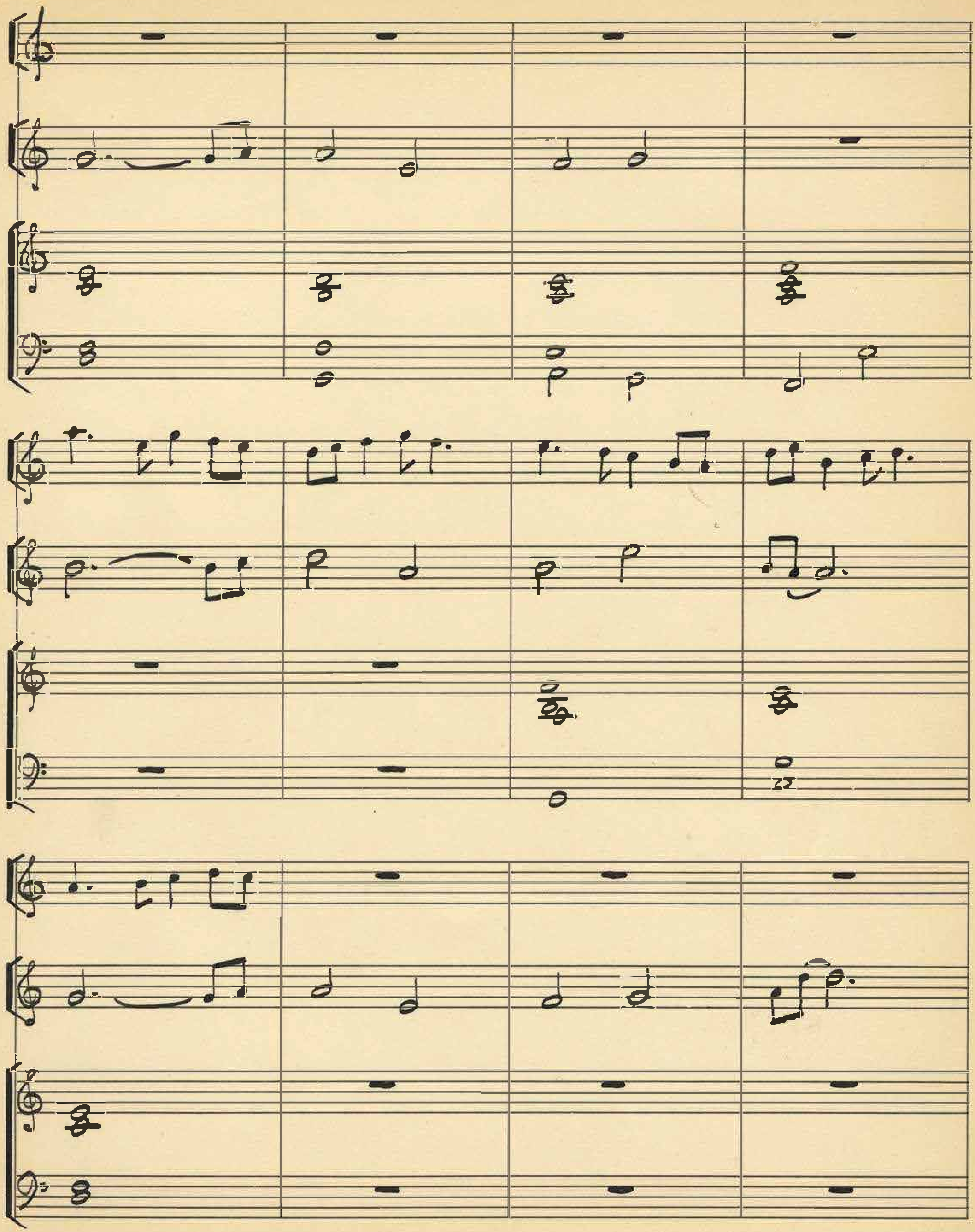


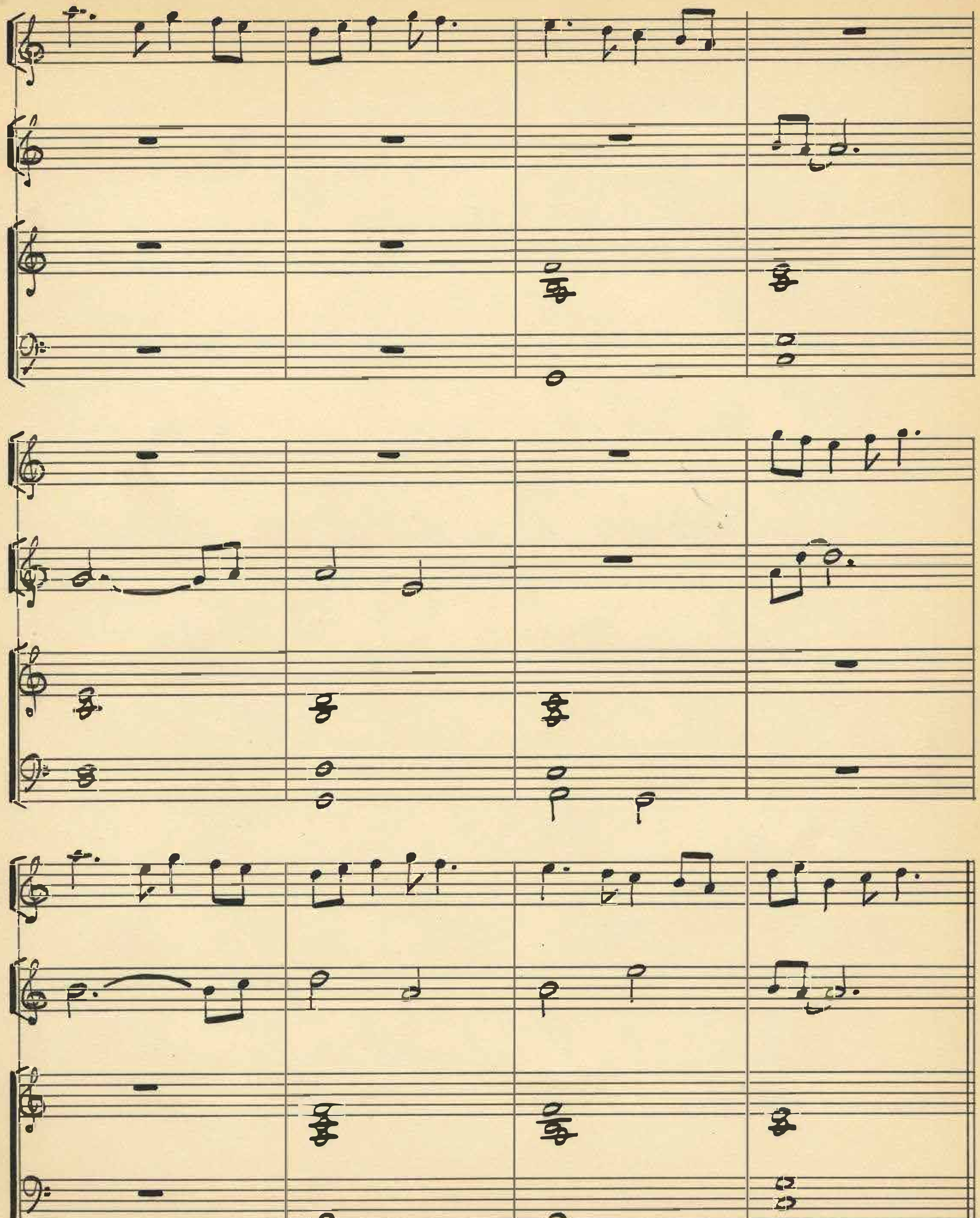












Lesson CLXXVI.

TWO-PART HARMONIZATION

The principle of writing a harmonic accompaniment to the duet of two contrapuntal parts consists of assigning harmonic consonances as chordal functions.

Every combination of two pitch-units producing a simultaneous consonance becomes a pair of chordal functions. This premise concerns all types of counterpoint and all types of harmonization.

Pitch-units producing dissonances are perceived through the auditory association as auxiliary and passing tones. Justification of the consonance as a pair of chordal functions gives meaning to the harmonic accompaniment.

Diatonic Harmonization of the Diatonic Two-Part Counterpoint.

Under the conditions imposed by Special Harmony, two-part counterpoint, which can be harmonized by the latter, must be constructed from seven-unit scales of the first group, not containing identical intonations.

As all three components must belong to one key, according to the definition of diatonic, the only types of counterpoint which can be diatonically harmonized are types I and II.

It is important for the composer to realize the modal versatility of relations which exist between the three components. As M_I may be written in any of the seven modes (d_0, d_1, d_2, d_3, d_4, d_5, d_6) of one scale, and so may M_{II} and the $H^{\rightarrow}$, the total number of modal variations for one scale is: $7^3 = 343$. This, of course, includes all the identical as well as non-identical combinations. Practically, however, this quantity must be somewhat limited, if we want to preserve the consonant relation between the P.A.'s of M_I and M_{II}.

It is important to remember that the number of seven-unit scales not containing identical units is 36. Therefore the total manifold of relations of M_I: M_{II}: $H^{\rightarrow}$ in the diatonic counterpoint of types I and II is:

$$343 \cdot 36 = 12{,}348.$$

Any given combination can be modified into a new system of intonations, i.e., into a new scale, by mere readjustment of the accidentals.

All the above quantities, naturally, do not include the attack-relations which have to be established for the harmonization.

As the attacks of $\frac{M_I}{M_{II}}$ are fixed groups, the only relation that is necessary to establish concerns $H^{\rightarrow}$. The most refined form of harmonization results from assigning each harmonic consonance to one H. If counter-

point contains many delayed resolutions of one dissonance, then the number of attacks of M_I is quite great and the changes of H are not as frequent. On the other hand, direct resolutions produce frequent chord changes. The assignment of two successive harmonic consonances to one H, amplifies the number of chords satisfying such a set, but at the same time neutralizes somewhat the character of $\overrightarrow{H}$. This technique, however, permits a greater variety of attack-relations between the three components.

We shall now proceed with the two-part diatonic harmonization.

Let us harmonize counterpoint type II, where $\frac{M_I}{M_{II}} = a$. In such a case all the harmonic intervals are consonances. Therefore we can have the following matching of attacks: $\frac{M_I = a}{\frac{M_{II} = a}{\overrightarrow{H} = a}}$ $\frac{M_I = 2a}{\frac{M_{II} = 2a}{\overrightarrow{H} = a}}$ $\frac{M_I = 3a}{\frac{M_{II} = 3a}{\overrightarrow{H} = a}}$ etc.

<u>Examples of Diatonic Harmonization of the Two-Part Counterpoint $\frac{M_I}{M_{II}} = a$.</u>

<u>Figure I.</u> (please see pages 4 and 5)

<u>Examples of Diatonic Harmonization of the Two-Part Counterpoint $\frac{M_I}{M_{II}} = \frac{3a}{a}$</u>

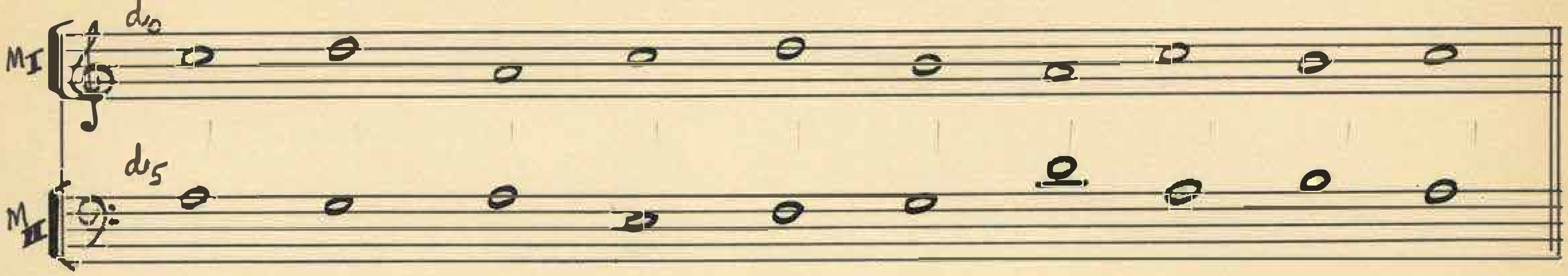
<u>Figure II.</u> (please see pages 5 and 6)

<u>Examples of Diatonic Harmonization of the Two-Part Counterpoint $\frac{M_I}{M_{II}} = \frac{4a}{a}$ and $\frac{M_I}{M_{II}} = \frac{6a}{a}$</u>

<u>Figure III.</u> (please see pages 6 and 7)

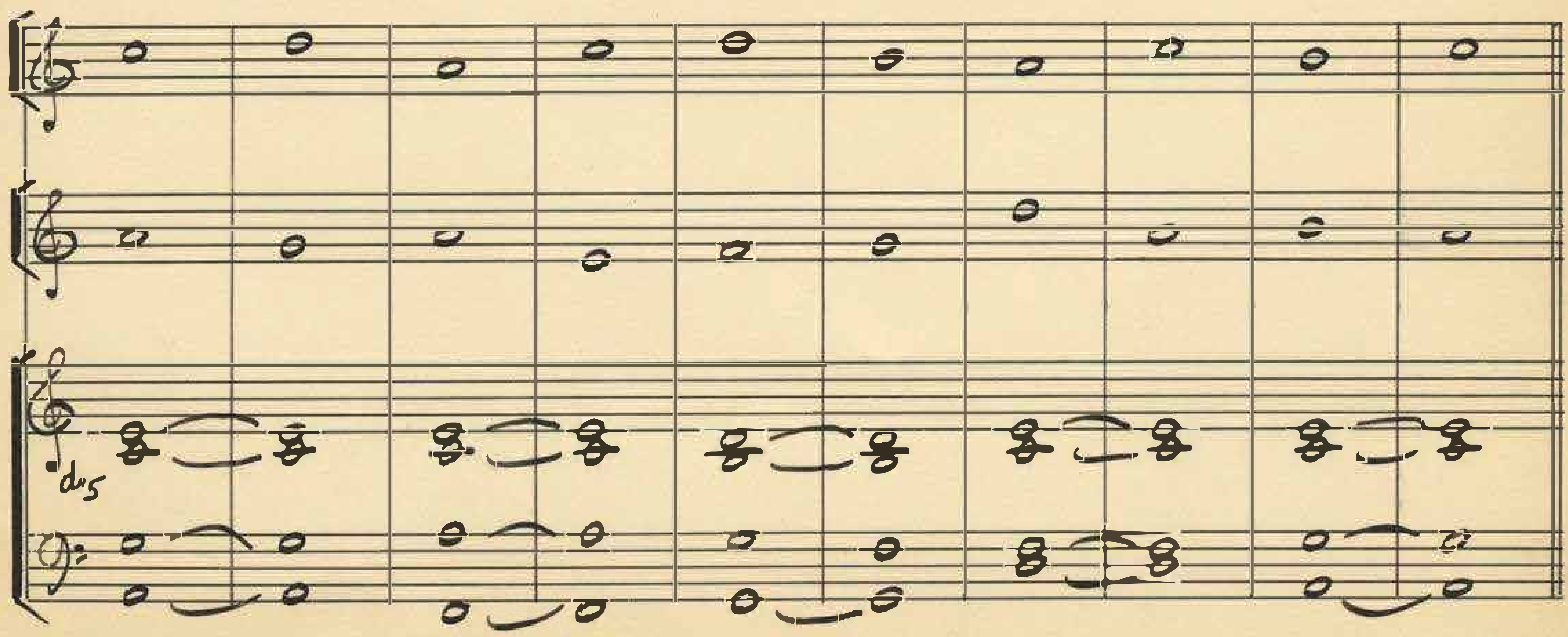
Figure I.

THEME:



HARMONIZATION (1)

HARMONIZATION (2)



HARMONIZATION (3)

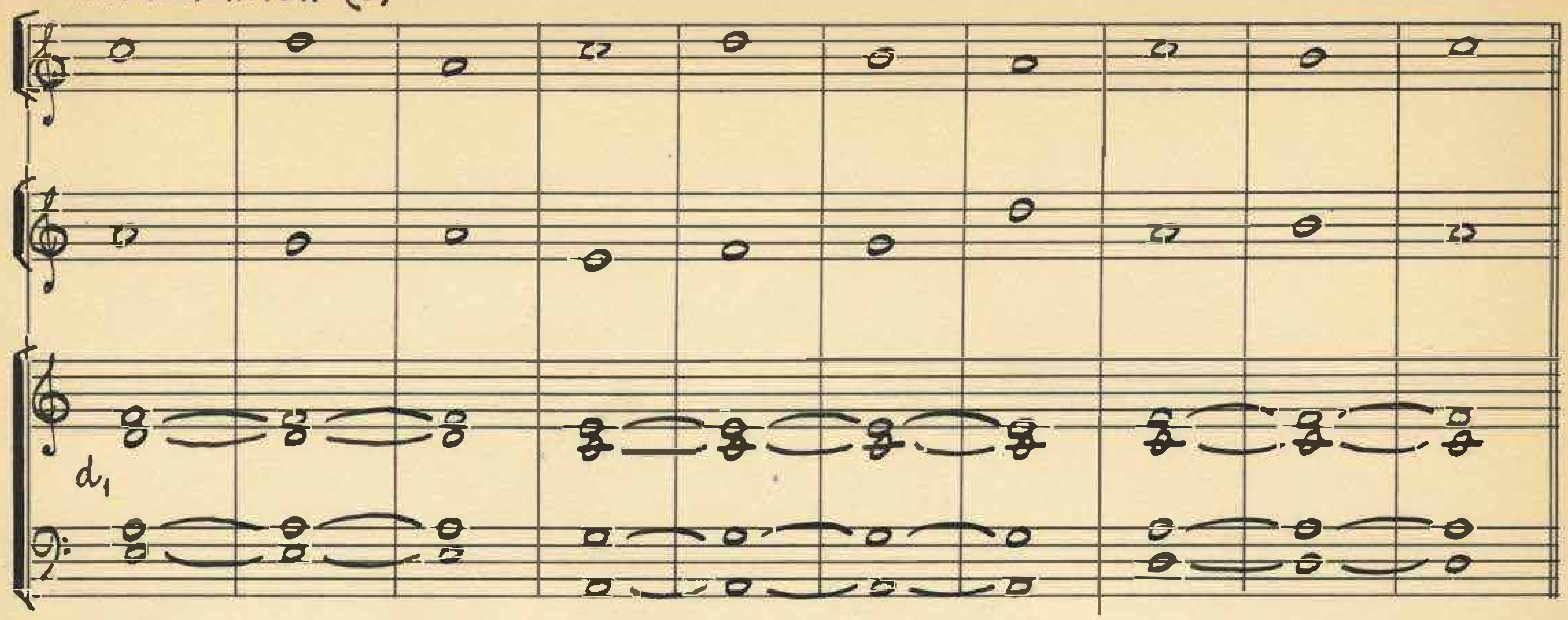
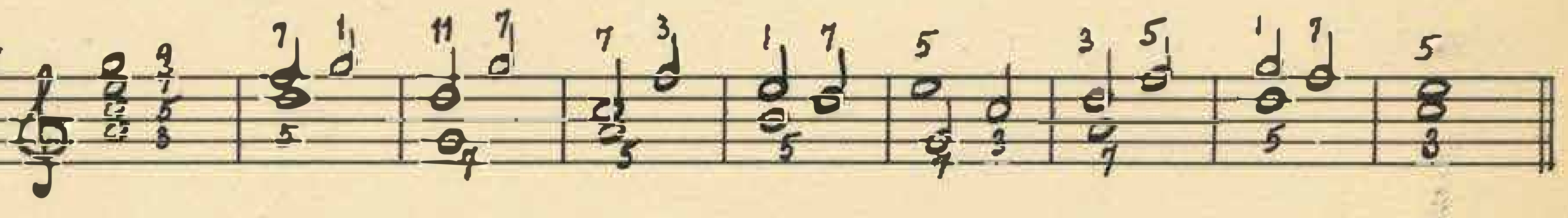


Figure II.

THEME:

HARMONIZATION (1):



HARMONIZATION (2)

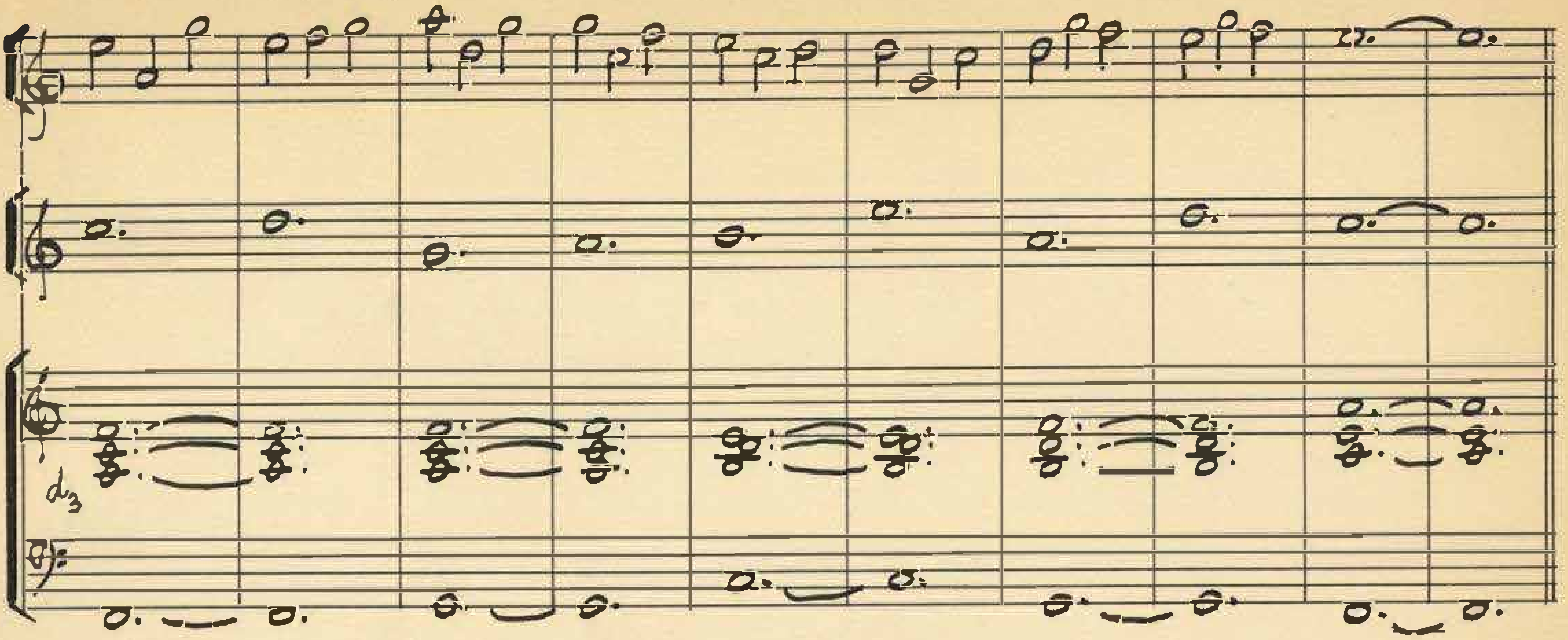
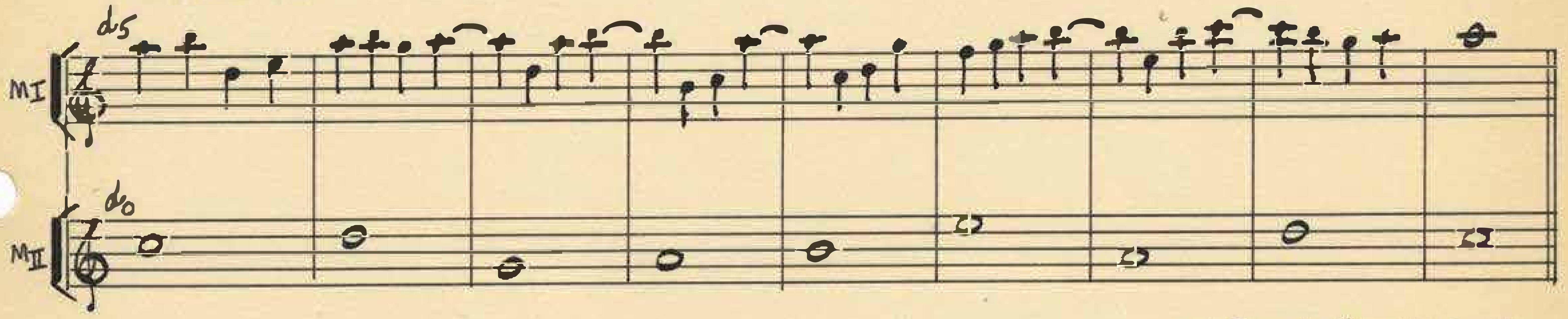


Figure III.

THEME: (1)



HARMONIZATION:

THEME: (2)

HARMONIZATION:

Lesson CLXXVII.

Chromatization of Harmony Accompanying Two-Part Diatonic Counterpoint (Types I and II).

A chromatic variation of the diatonic harmony accomp_anying two-part counterpoint can be obtained by means of auxiliary and passing chromatic tones. Of course such altered tones shall not conflict in any way with the two melodies.

For our example we shall take the two-part counterpoint diatonically harmonized from Figure III (2).

Example of the Chromatization of Harmonic Accompaniment

Figure IV.

(please see page 9)

Diatonic Harmonization of the Chromatic Counterpoint Whose Origin is Diatonic (Types I and II)

The principle of this form of harmonization consists of assigning the diatonic consonances as chordal functions. Chromatic consonances as well as all other forms of harmonic intervalsshall be neglected.

The quantity of successive consonances corresponding to one H is optional. It is practical to make T or 2T, or 3T correspond to one H.

When harmonizing a chromatic counterpoint, whose diatonic original is known, one can assign chordal

Figure IV.

functions directly from the diatonic original. This measure obviously eliminates any possible confusion of the diatonic and the chromatic consonances.

We shall now harmonize a duet where both parts are chromatic. The theme is taken from Figure XXIV of the Two-Part Counterpoint. For clarity's sake, we shall write out both the original and the chromatized version. We shall choose the following relationship between $H^{\rightarrow}$ and $T^{\rightarrow}$:

$$H^{\rightarrow} T^{\rightarrow} = HT + H2T + HT + HT + HT + H2T + HT$$

which is a modified version of the $r_{3\div2}$, and which permits to demonstrate the diversified forms of attacks groups of M_{I} and M_{II} in relation to $H^{\rightarrow}$.

<u>Example of Diatonic Harmonization of</u>

<u>the Chromatic Counterpoint</u>

<u>Figure V.</u>

(please see page 11)

When the diatonic origin of chromatic counterpoint is unknown, the analysis of diatonic consonances must precede the planning of harmonization.

Figure V.

ORIGINAL.

CHROMATIC VARIATION.

Lesson CLXXVIII.

Symmetric Harmonization of the Diatonic Two-Part Counterpoint (Types I, II, III and IV).

The principle of symmetric harmonization of the two-part counterpoint consists of assigning all harmonic intervals as chordal functions.

The fewer attacks of M_I and M_{II} correspond to one H, the easier it is to perform such harmonization by means of one $\Sigma 13$. When a considerable number of attacks (even in one of the two melodies) corresponds to one H, it becomes necessary to introduce two, and sometimes three $\Sigma 13$. The forms of the latter should vary only slightly, serving the only purpose of rectifying the non-corresponding pitch-unit. For instance, when using $\Sigma 13$ XIII as Σ_1, correction of the eleventh to f♮ gives satisfactory solution for most cases. Thus, Σ_2 in this instance differs from Σ_1 only with respect to 11.

The selection of the original $\Sigma 13$ is a matter of harmonic character. For example, the use of $\Sigma 13$ XIII attributes to music a definitely Ravelian quality. However, harmonic quality still remains virgin territory awaiting the composer's exploration. Most of the 36 forms of the $\Sigma 13$ have not been utilized.

Whether counterpoint belongs to types I and II, or to types III and IV, it does not give any clue to any particular $\Sigma 13$. And whereas symmetric

harmonization of the counterpoint of types I and II is a luxury, it is a bare necessity for types III and IV, as the latter correlate two different key-axes. The fact that two different keys with identical or with non-identical scales can be united by one chord is of particular importance. This is so because the quality of a selected Σ 13 is capable of influencing the two melodies. The ear in our musical civilization is so much conditioned by harmony, that most of our listeners have lost the ability of enjoying melodic line per se. And if the ear of an average music-lover can relate one diatonic melody to some chord progression, the harmonic association of two melodies belonging to two different keys becomes impossible. Therefore the <u>role of a harmonic master-structure</u> (Σ 13 in this case) is one of a <u>synthesizer.</u>

The simplest way to assign harmonic functions is by relating the latter to consonances first.

The master-structure used in the following harmonizations is Σ 13 XIII.

<u>Symmetric Harmonization of the Diatonic</u>

<u>Two-Part Counterpoint of Types I and II.</u>

<u>Figure VI.</u>

(please see next page)


(1)
(2)
(3) CHROMATIC VARIATION OF HARMONY. (2)

Chromatic variation of $H^{\rightarrow}$ in the above example is obtained through the usual technique: the insertion of passing and auxiliary units.

Symmetric Harmonization of the Diatonic Two-Part Counterpoint of Types III and IV.

Figure VII.

(please see page 16)

Symmetric Harmonization of the Chromatic Two-Part Counterpoint Whose Origin is Diatonic (Types I, II, III and IV).

The principle of symmetric harmonization of the chromatic two-part counterpoint consists of assigning all the diatonic pitch-units of both melodies as chordal functions of the master-structure (Σ 13) and neglecting all the chromatic pitch-units, as not belonging to the scale. It does not matter whether the chromatic units belong to the master-structure or not. When the diatonic original of the two-part counterpoint is unknown, the diatonic units of both melodies should be detected first.

Figure VIII.

(please see page 17)

Counterpoint executed in symmetric scales of the Third and the Fourth Group can be harmonized by

Figure VII.

(1)

(2) CHROMATIC VARIATION OF HARMONY (1)

Figure VIII.

ORIGINAL (1)

CHROMATIC VARIATION HARMONIZED (1)

ORIGINAL (2)

CHROMATIC VARIATION HARMONIZED (2)

means of a symmetric master-structure. This master-structure is independent of the system of symmetry of the pitch-scales involved. As in the previous cases, all units corresponding to one H must belong to one Σ 13.

After the harmonization is performed, it may be subjected, if desirable, to chromatic variation.

Symmetric Harmonization of the
Symmetric Two-Part Counterpoint.

Figure IX.

(please see page 19)

All forms of contrapuntal continuity as well as complete compositions in the form of canons and fugues can be harmonized accordingly to this technique. Any of the above described correspondences between counterpoint and harmony can be established by the composer. One should remember that overloading harmonic accompaniments is more a sin than a virtue. For this reason the technique of variable density should receive utmost consideration.

Figure IX.

THEME:

HARMONIZATION

CHROMATIC VARIATION OF HARMONY.

Lesson CLXXIX.

Ostinato

Forms of ostinato or ground motion have been known since time immemorial. They appear in different folk and traditional music as a fundamental form of improvisation around a given theme. The characteristic of ostinato (literally obstinate) is a continuous repetition of a certain thematic group, which may be either rhythm, or melody, or harmony. For example, the dance beat of 4/4 in a fox-trot is one of such fundamental forms of ostinato. As a matter of fact, a rhythmic ostinato is ever present in all the developments in classical symphonies. Take, for example, Beethoven's Fifth Symphony, the first motif of it consisting of 4 notes, and follow it up through the development (middle section of the first movement). The motif, rhythmically the same, changes its forms of intonation either melodically or in the form of accompanying harmony.

Repetitions of groups of chords, as well as repetitions of melodic fragments accompanied by continuously changing chords, are forms of ostinato. Ostinato is one of the traditional forms of thematic growth and, as such, is very well known in the form of ciaconna and passacaglia. In many Irish jigs, ostinato appears in forms of pedal point as well as in repetitious melodic fragments. When

portions of the same melody appear in succession, being harmonized every time anew, (which may be found even in such works as Chopin's Mazurkas,) we have a case of ostinato.

I. Melodic Ostinato

(Basso Ostinato)

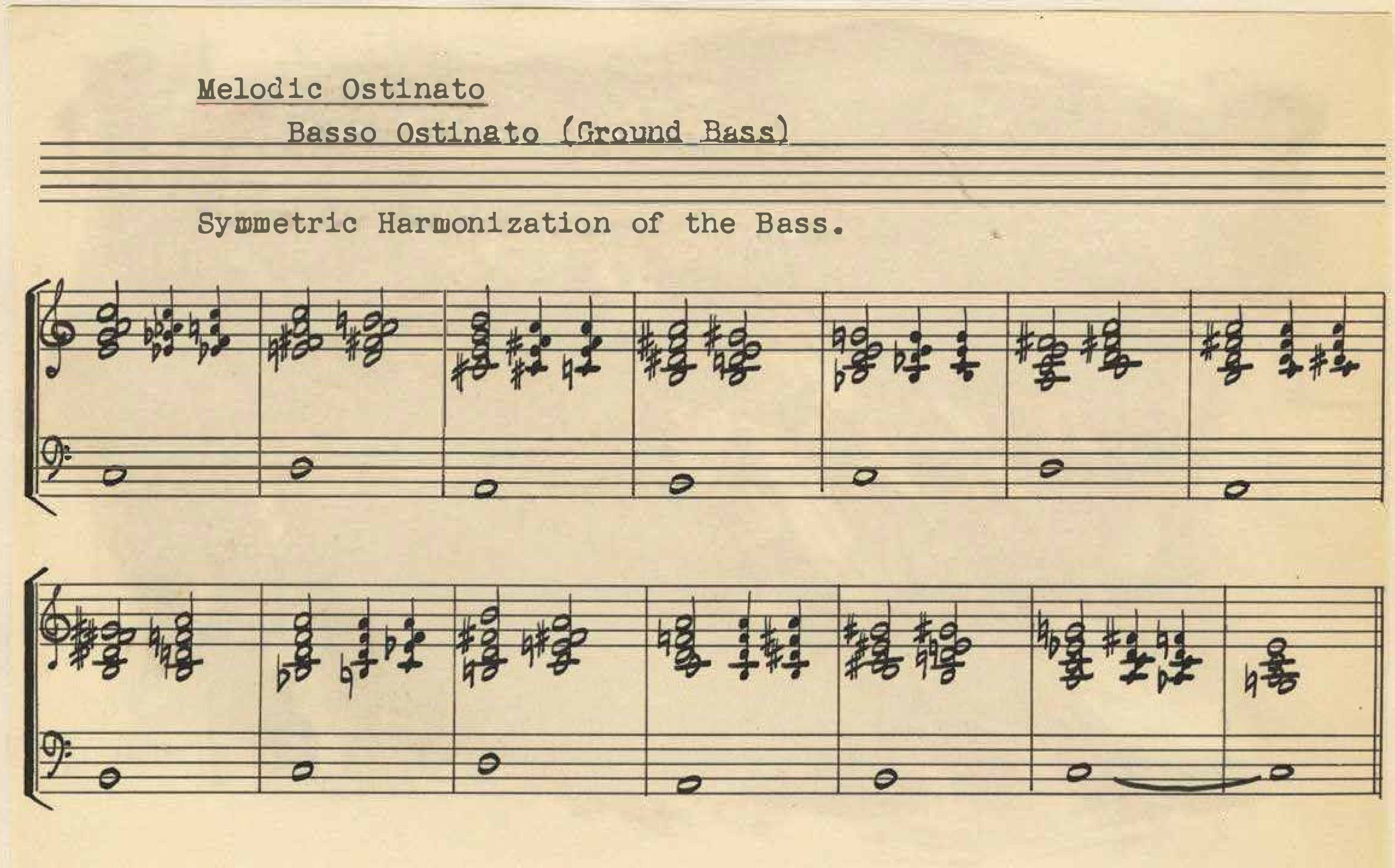
Melodic ostinato, better known under the name of "Ground Bass", is a harmonization of an ever-repeating melody with continuously changing chords. Ostinato groups produce one uninterrupted continuity where the recurrence of the bass form produces unity, and the accompanying harmony - variety. All forms of harmonization can be applied to the continuously repeating melody, and regardless as to whether it appears in the bass or in any of the middle voices, or in the upper voice (above harmony).

As every harmonic setting of chords is subject to vertical permutations, a basso ostinato can be transformed into tenor, or alto, or soprano ostinato, i.e., it may appear in any desirable voice and in any desirable sequence after the harmonization has been completed.

In the following example the ostinato of the theme is a melody in whole notes in the bass (the first four bars), after which it repeats itself two more times. The form of harmonization is symmetric in this case, though it could be diatonic or any of the chromatic forms. This device can be used as a form of thematic development,

and in arranging for the purpose of constructing introductions or transitions, as any characteristic melodic pattern can be converted into basso ostinato either with the preservation of its original rhythm or in an entirely new setting. *

Figure I.



*See: Arensky's "Basso Ostinato" for Piano.

II. Harmonic Ostinato

Harmonic Ostinato may be also called, by analogy, "ground harmony". It consists of the repetition of a group of chords in relation to which a continuously changing melody is evolved. This form of ostinato is the one J.S. Bach employed in his D-minor "Ciaconna" for Violin, besides numerous other compositions by Bach and other composers. Among my students, a successful use of this device occurred in an exercise made by George Gershwin, and which later, at my suggestion, was put into the musical comedy, "Let 'Em Eat Cake", of which it became the hit song ("Mine").

This form of ostinato can be applied to any type of harmonic progressions. The technical procedure is exactly the opposite of the first one. In this case we deal with melodization of harmony. As in the previous case, the melody evolved against chords may be transferred to a different position in relation to chord by means of vertical permutation. Naturally, not every melody will be equally as good under such conditions if it appears in the bass and in the soprano, as the chordal functions represented by melody may be more advantageous for an upper part than for the lower, or vice versa.

In the following example, the harmonic theme of ostinato emphasizes four different chords (the first two bars), and is based on a Σ 13 XIII. The melody

evolves through the principle of symmetric melodization forming its axis points in relation to the chord structure itself. The main resource of variety is the manifold of melodic forms.

Figure II.

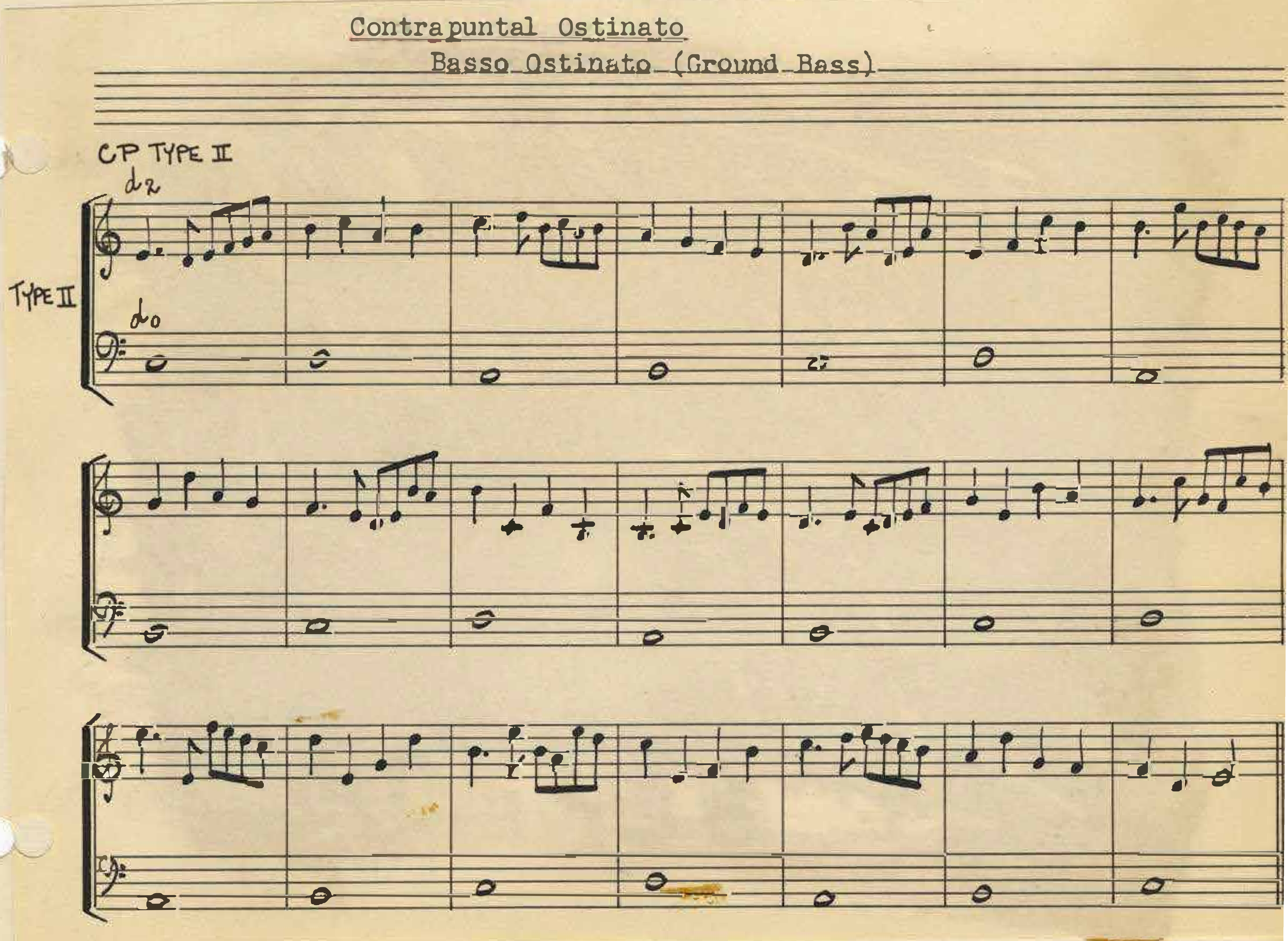
III. Contrapuntal Ostinato.

The form of contrapuntal ostinato is well known through the works of old masters, and was usually evolved to a melody known as "cantus firmus". If a C.F. repeats itself continuously a number of times while the contrapuntal part or parts evolve in relation to it,

producing different relations with every appearance of the C.F., we have a contrapuntal ostinato.

In the following example, the theme of ostinato is taken from Figure I, and the accompanying counterpoint is evolved through Type II, adhering to a rhythmic ostinato as well (except for a few intentional permutations). Naturally both voices can be exchanged as well as subjected to any of the variations through geometrical positions (a), (b), (c), and (d) .

Figure III.



Likewise, a counterpoint can be evolved to the soprano voice through the use of the same principle. In Figure IV, the same theme is employed except that it is altered rhythmically, and the counterpoint, in its rhythmic setting, produces a constant interference against the C.F., as it consists of a 3-bar group. The harmonic setting of this example is in Type III: the C.F. is in natural C major, and the counterpoint is in natural $A^{\flat}$ major.

Figure IV.

The last two forms of ostinato are extremely adaptable in all cases when it is desirable to repeat one motif and yet introduce variety into an obligato. These characteristics make the above described device extremely useful for introductions, transitions and codas, when applied to arranging.

J O S E P H   S C H I L L I N G E R

C O R R E S P O N D E N C E   C O U R S E

With: Dr. Jerome Gross   Subject: Music

Lesson CLXXX.

INSTRUMENTAL FORMS OF MELODY AND HARMONY

The meaning of Instrumental Form implies a modification of the original which renders the latter fit for execution on an instrument. Instrumental can be defined as an applied form of the pure. Depending on the degree of virtuosity which is to be expected from the performers, instrumental forms may be applied to vocal music as well.

The main technical characteristic of the instrumental (i.e., of applied versus pure) form is the development of the quantities (multiplication) and forms of attacks from the original attack. This branch will be concerned only with the first, i.e., with quantities and their uses in composition, leaving the second, i.e., the forms of attacks (such as durable, abrupt, bouncing, oscillating, etc.), to the branch of Orchestration.

Multiplication of attacks can be applied directly to single pitch-units as well as to pitch-assemblages. The quantity of the instrumental forms is dependent upon the quantity of pitch-units in an assemblage.

When the quantity of pitch-units (parts) in an assemblage is scarce, the number of instrumental forms is low. When the number of pitch-units (parts) in an assemblage is abundant, the number of instrumental forms is high. The latter permits to accomplish greater variety in a composition, insofar as its instrumental aspect is concerned.

The scarcity of instrumental forms derived from one pitch-unit (part) often makes it compelling to resort to couplings. By addition of one coupling to one part we achieve a two-part setting, with all its instrumental implications. Likewise, the addition of two couplings to one part transforms the latter into a three-part assemblage, etc.

This branch consists of an exhaustive study of all forms of arpeggio and their applications in the field of melody, harmony and correlated melodies.

Nomenclature:

Σ --- Score (Group of instrumental strata)

S --- Stratum (instrumental stratum)

p --- part (function, coupling)

a --- attack

Preliminary Data:

(1) $p = a$; $p = 2a$; . . . $p = na$

(2) $S = p$; $S = 2p$; . . . $S = np$

(3) $\Sigma = S$; $\Sigma = 2S$; . . . $\Sigma = nS$

Sources of Instrumental Forms

(a) Multiplication of S is achieved by 1 : 2 : 4 : 8 : ... ratio (i.e., by the octaves)

(b) Multiplication of p in S is achieved by coupling or by harmonization. It is applicable to melody (p), correlated melodies (2p, ... np) and harmony (2p...4p). The material for p is in the Theory of Pitch Scales and the Theory of Melody. The material for 2p, ... np acting as melodies is in the Theory of Correlated Melodies (Counterpoint). The material for 2p,...np acting as parts of harmony is in the Special Theory of Harmony and in the General Theory of Harmony.

(c) Multiplication of a is achieved by repetition and sequence of p's (arpeggio).

(d) Different S's and different p's, as correlated melodies of Σ may have independent instrumental forms.

Definition of the Instrumental Forms:

I. (a) Instrumental Forms of Melody: I(M = p): repetition of pitch-units represented by the duration-group and expressed through its common denominator. The number of a equals the number of t.

$$\text{If } \frac{1}{nt} = nt, \text{ then } nt = na$$

Rhythmic composition of durations assigned to each attack.

(b) Instrumental Forms of Melody: I (M = np): repetition of pitch-units (p_I) and their couplings (p_{II}, p_{III}, ... p_N) and transition (sequence) from one

p to another, represented by the duration group and expressed through its common denominator. Instrumental groups of p's consisting of repetitions and sequences are subject to permutations.

(α) Instrumental Forms of the Simultaneous Groups of Melody:

$$M = \frac{p_{II}}{p_I};\ \frac{p_I}{p_{II}};\ \frac{p_{III}}{\frac{p_I}{p_{II}}};\ \frac{p_{II}}{\frac{p_I}{p_{III}}};\ \frac{p_I}{\frac{p_{II}}{p_{III}}};\ \frac{p_I}{\frac{p_{III}}{p_{II}}};\ \frac{p_{III}}{\frac{p_{II}}{p_I}};\ \frac{p_{II}}{\frac{p_{III}}{p_I}};\ldots$$

(β) Instrumental Forms of the Sequent Groups of Melody:

$$M = p_I + p_{II};\ p_{II} + p_I;\ p_I + p_{II} + p_{III};\ p_I + p_{III} + p_{II};\ p_{III} + p_I + p_{II};\ p_{II} + p_I + p_{III};\ p_{II} + p_{III} + p_I;\ p_{III} + p_{II} + p_I.$$

$$M = p_I + p_{II} + p_I;\ p_I + p_{II} + p_{III} + p_I;\ p_I + p_{II} + p_{III} + p_{II};\ p_I + p_{II} + p_{III} + p_{III}.$$

(γ) Instrumental Forms of the Combined Groups of Melody:

$$M = \frac{p_{II}}{p_I} + \frac{p_{III}}{p_I} + \frac{p_{III}}{p_{II}};\ \frac{p_{II}}{p_I} + \frac{p_{III}}{p_I} + \frac{p_{IV}}{p_I} + \frac{p_{III}}{p_{II}} + \frac{p_{IV}}{p_{II}} + \frac{p_{IV}}{p_{III}};$$

$$M = \frac{p_{III}}{\frac{p_{II}}{p_I}} + \frac{p_{IV}}{\frac{p_{II}}{p_I}} + \frac{p_{IV}}{\frac{p_{III}}{p_I}} + \frac{p_{IV}}{\frac{p_{III}}{p_{II}}};\ \ldots$$

II. Instrumental Forms of Correlated Melodies:

(a) $I\left(\frac{M_{II} = p}{M_I = p}\right)$: correlation of instrumental forms of the two uncoupled melodies (M_I and M_{II}) by means of correlating their a's.

M_I (nt = na); M_{II} (nt = 2na; 3na; ... mna)

$$\frac{M_{II}\ (t = a)}{M_I\ (t = 2a)};\ \frac{M_{II}\ (t = 2a)}{M_I\ (t = a)};\ \frac{M_{II}\ (t = a)}{M_I\ (t = 3a)};\ \frac{M_{II}\ (t = 3a)}{M_I\ (t = a)}$$

$$\frac{M_{II}\ (t = 2a)}{M_I\ (t = 3a)};\ \frac{M_{II}\ (t = 3a)}{M_I\ (t = 2a)};\ \frac{M_{II}\ (t = a)}{M_I\ (t = 4a)};\ \frac{M_{II}\ (t = 4a)}{M_I\ (t = a)};$$

$$\frac{M_{II}\ (t = 2a)}{M_I\ (t = 4a)};\ \frac{M_{II}\ (t = 4a)}{M_I\ (t = 2a)};\ \frac{M_{II}\ (t = 3a)}{M_I\ (t = 4a)};\ \frac{M_{II}\ (t = 4a)}{M_I\ (t = 3a)};\ \ldots$$

$$\ldots\ \frac{M_{II}\ (t = na)}{M_I\ (t = ma)}$$

(b) $I\left(\frac{M_{II} = np}{M_I = mp}\right)$: this form corresponds to combinations of (α), (β) and (γ) of I (b).

$$\frac{M_{II}\ (\alpha)}{M_I\ (\alpha)};\ \frac{M_{II}\ (\alpha)}{M_I\ (\beta)};\ \frac{M_{II}\ (\beta)}{M_I\ (\alpha)};\ \frac{M_{II}\ (\beta)}{M_I\ (\beta)};$$

$$\frac{M_{II}\ (\alpha)}{M_I\ (\gamma)};\ \frac{M_{II}\ (\gamma)}{M_I\ (\alpha)};\ \frac{M_{II}\ (\beta)}{M_I\ (\gamma)};\ \frac{M_{II}\ (\gamma)}{M_I\ (\beta)};\ \frac{M_{II}\ (\gamma)}{M_I\ (\gamma)}.$$

III. Instrumental Forms of Harmony:

I (S = p, 2p, 3p, 4p): this corresponds to one part harmony, which is the equivalent of M; two-part harmony,

which is the equivalent of two correlated uncoupled melodies; three-part harmony, which is the equivalent of three correlated uncoupled melodies; four-part harmony, which is the equivalent of four correlated uncoupled melodies.

The source of Harmony can be the Theory of Pitch Scales, Special Theory of Harmony and General Theory of Harmony. Parts (p's) in their simultaneous and sequent groupings correspond to a, b, c, d.

$p_{I} = a; \; p_{II} = b; \; p_{III} = c; \; p_{IV} = d.$

<u>Instrumental Forms of S = p.</u>

Material:

(a) melody;

(b) any one of the correlated melodies;

(c) one-part harmony;

(d) harmonic form of one unit scale;

(e) one part of any harmony.

I = a; 2a; 3a; ma; A var.

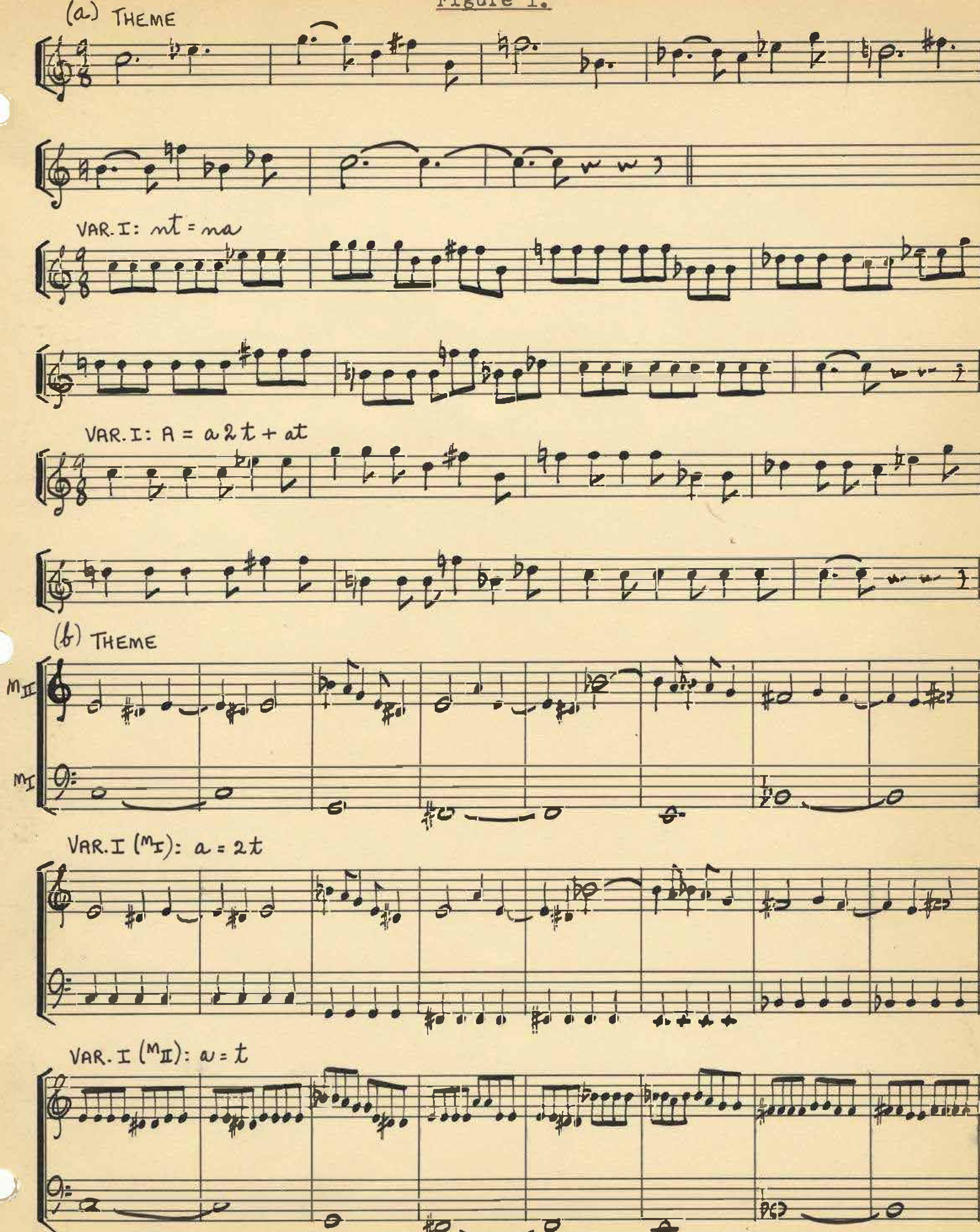
nt = na

~~oll-ooll-oooll-ooooll-oooooll-ooooooll-ooooooooll~~

<u>Figure I.</u>

(please see pages 7 and 8)

Figure I.

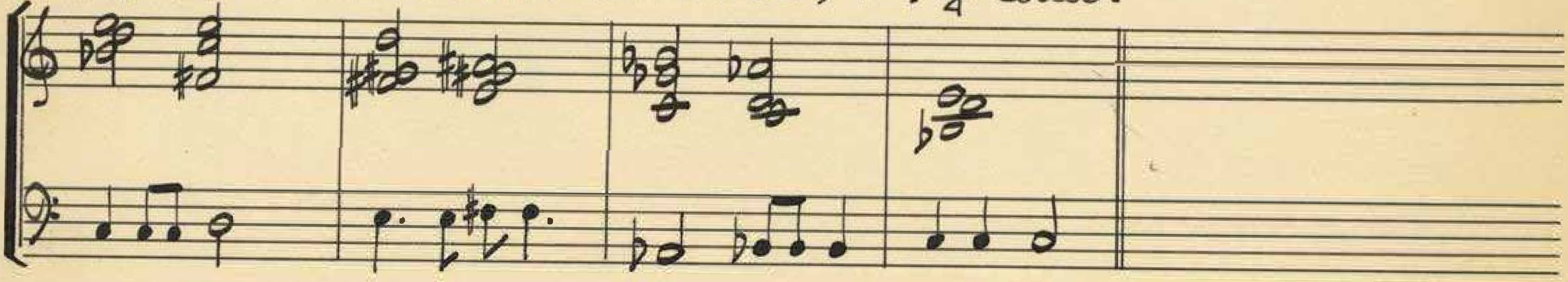

(a) THEME
VAR. I: nt = na
VAR. I: A = a 2t + at
(b) THEME
MII
MI
VAR. I (MI): a = 2t
VAR. I (MII): a = t

VAR. I $\left(\frac{M II}{M I}\right)$: $\frac{a = t}{a = 2t}$

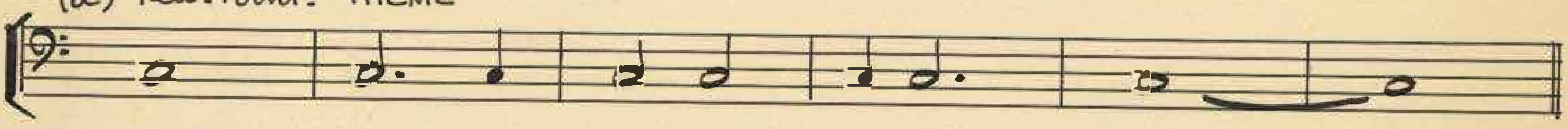
(c) THEME

VAR. I (B): a = t

VAR I (B): A = 3a + a + 2a + 2a + a + 3a ; T'': $\frac{4}{4}$ series.



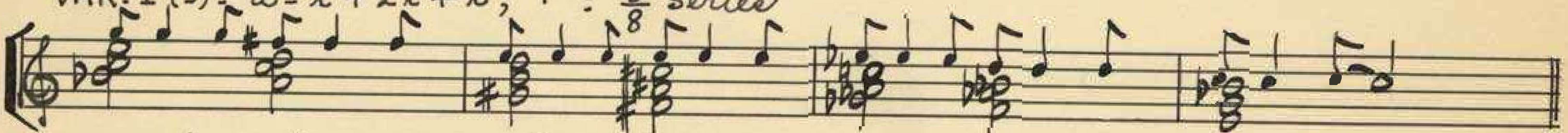
(d) Ped. Point. THEME



VAR. I (p): a = t

(e) THEME

VAR. I (S): a = t + 2t + t; T'': $\frac{8}{8}$ series



VAR. I (ABTS): a = t ; T'' $\frac{8}{8}$ series

Lesson CLXXXI.

General Classification of I (S = 2p).

(Table of the combinations of attacks for a and b)

A = a; 2a; 3a; 4a; 5a; 6a; 7a; 8a; 12 a.

The following is a complete table of all forms of I (S = 2p). It includes all the combinations and permutations for 2, 3, 4, 5, 6, 7, 8 and 12 attacks.

A = 2a; a + b.

$$P_2 = 2! = 2$$

Total in general permutations: 2

Total in circular permutations: 2

A = 3a; 2a + b; a + 2b.

$$P_3 = \frac{3!}{2!} = \frac{6}{2} = 3$$

Each of the above 2 permutations of the coefficients has 3 general permutations.

Total: 3·2 = 6

The total number of cases A = 3a

General permutations: 6

Circular permutations: 6

A = 4a

Forms of the distribution of coefficients:

4 = 1+3; 2+2

A = a + 3b; 3a + b.

$$P_4 = \frac{4!}{3!} = \frac{24}{6} = 4$$

Each of the above 2 permutations of the first form of distribution of the coefficients of recurrence has 4 general permutations.

Total: $4 \cdot 2 = 8$

$A = 2a + 2b$

$$P_4 = \frac{4!}{2!\ 2!} = \frac{24}{2 \cdot 2} = 6$$

The above invariant form of distribution has 6 general permutations.

The total number of cases: $A = 4a$

General permutations: $8 + 6 = 14$

Circular permutations: $4 \cdot 3 = 12$

$A = 5a$

Forms of the distribution of coefficients:

$5 = 1+4;\ 2+3.$

$A = a + 4b;\ 4a + b$

$$P_5 = \frac{5!}{4!} = \frac{120}{24} = 5$$

Each of the above 2 permutations of the first form of distribution has 5 general permutations.

Total: $5 \cdot 2 = 10$

$A = 2a + 3b;\ 3a + 2b.$

$$P_5 = \frac{5!}{2!\ 3!} = \frac{120}{2 \cdot 6} = 10$$

Each of the above 2 permutations of the second

form of distribution has 10 general permutations.

Total: 10·2 = 20

The total number of cases: A = 5a

General permutations: 10 + 20 = 30

Circular permutations: 5·4 = 20

A = 6a

Forms of the distribution of coefficients:

6 = 1+5; 2+4; 3+3 .

A = a + 5b; 5a + b.

$$P_6 = \frac{6!}{5!} = \frac{720}{120} = 6$$

Each of the above 2 permutations of the first form of distribution has 6 general permutations.

Total: 6·2 = 12

A = 2a + 4b; 4a + 2b.

$$P_6 = \frac{6!}{2!\ 4!} \quad \frac{720}{2 \cdot 24} = 15$$

Each of the above 2 permutations of the second form of distribution has 15 general permutations.

Total: 15·2 = 30

A = 3a + 3b

$$P_6 = \frac{6!}{3!\ 3!} = \frac{720}{6 \cdot 6} = 20$$

The above invariant (third) form of distribution has 20 general permutations.

The total number of cases: A = 6a

General permutations: 12 + 30 + 20 = 62

Circular permutations: 6·5 = 30

A = 7a

Forms of the distribution of coefficients:

7 = 1+6; 2+5; 3+4 .

A = a + 6b; 6a + b.

$$P_7 = \frac{7!}{6!} = \frac{5040}{720} = 7$$

Each of the above 2 permutations of the first form of distribution has 7 general permutations.

Total: 7·2 = 14

A = 2a + 5b; 5a + 2b .

$$P_7 = \frac{7!}{2!\ 5!} = \frac{5040}{2 \cdot 120} = 21$$

Each of the above 2 permutations of the second form of distribution has 21 general permutations.

Total: 21·2 = 42

A = 3a + 4b; 4a + 3b.

$$P_7 = \frac{7!}{3!\ 4!} = \frac{5040}{6 \cdot 24} = 35$$

Each of the above 2 permutations of the third form of distribution has 35 general permutations.

Total: 35·2 = 70

The total number of cases: A = 7a

General permutations: 14 + 42 + 70 = 126

Circular permutations: 7·6 = 42

A = 8a

Forms of the distribution of coefficients:

8 = 1+7; 2+6; 3+5; 4+4 .

A = a + 7b; 7a + b .

$$P_8 = \frac{8!}{7!} = \frac{40,320}{5,040} = 8$$

Each of the above 2 permutations of the first form of distribution has 8 general permutations.

Total: 8·2 = 16

A = 2a + 6b; 6a + 2b .

$$P_8 = \frac{8!}{2!\ 6!} = \frac{40,320}{2 \cdot 720} = 28$$

Each of the above 2 permutations of the second form of distribution has 28 general permutations.

Total: 28·2 = 56

A = 3a + 5b; 5a + 3b .

$$P_8 = \frac{8!}{3!\ 5!} = \frac{40,320}{6 \cdot 120} = 56$$

Each of the above 2 permutations of the third form of distribution has 56 general permutations.

Total: 56·2 = 112

A = 4a + 4b

$$P_8 = \frac{8!}{4!\ 4!} = \frac{40,320}{24 \cdot 24} = 70$$

The above invariant (fourth) form of distribution has 70 general permutations.

The total number of cases: A = 8a

General permutations: 16 + 56 + 112 + 70 = 254

Circular permutations: 8·7 = 56

A = 12a

Forms of the distribution of coefficients:

12 = 1+11; 2+10; 3+9; 4+8; 5+7; 6+6

A = a + 11b; 11a + b.

$$P_{12} = \frac{12!}{11!} = \frac{479,001,600}{39,916,800} = 12$$

Each of the above 2 permutations of the first form of distribution has 12 general permutations.

Total: 12·2 = 24

A = 2a + 10b; 10a + 2b.

$$P_{12} = \frac{12!}{2!\ 10!} = \frac{479,001,600}{2\cdot 3,628,800} = 66$$

Each of the above 2 permutations of the second form of distribution has 66 general permutations.

Total: 66·2 = 132

A = 3a + 9b; 9a + 3b.

$$P_{12} = \frac{12!}{3!\ 9!} = \frac{479,001,600}{6\cdot 362,880} = 220$$

Each of the above 2 permutations of the third form of distribution has 220 general permutations.

Total: 220·2 = 440

A = 4a + 8b; 8a + 4b.

$$P_{12} = \frac{12!}{4!\ 8!} = \frac{479,001,600}{24 \cdot 40,320} = 495$$

Each of the above 2 permutations of the fourth form of distribution has 495 general permutations.

Total: 495·2 = 990

A = 5a + 7b; 7a + 5b.

$$P_{12} = \frac{12!}{5!\ 7!} = \frac{479,001,600}{120 \cdot 5,040} = 792$$

Each of the ab_ove 2 permutations of the fifth form of distribution has 792 general permutations.

Total: 792·2 = 1584

A = 6a + 6b

$$P_{12} = \frac{12!}{6!\ 6!} = \frac{479,001,600}{720 \cdot 720} = 924$$

The above invariant (sixth) form of distribution has 924 general permutations.

The total number of Cases: A = 12a

General permutations: 24 + 132 + 440 + 990 + + 1584 + 924 = 4094.

Circular permutations: 12·11 = 132

Lesson CLXXXII.

Figure II.

The interval of octave can be changed to any other interval. For the groups with more than 6 attacks only circular permutations are included.

(please see pages 17-22)

Figure III.

Examples of the polynomial attack-groups (coefficients of recurrence).

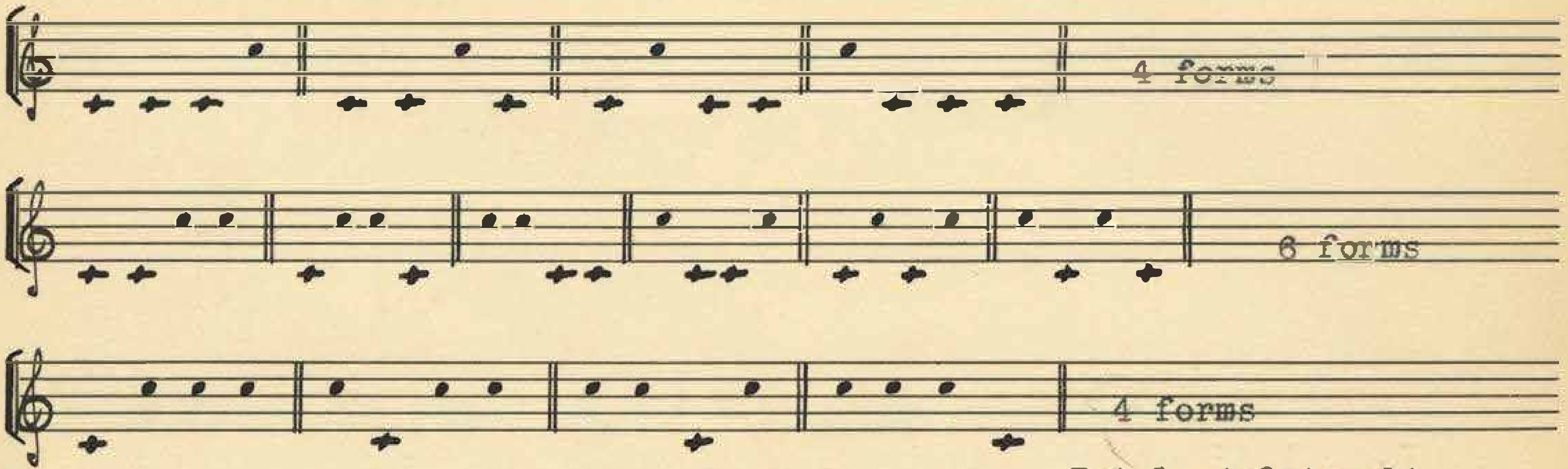
(please see page 22)

Figure II.

A = a   A = 2a; 2 forms   A = 3a: 2a+b; a+2b. 2 combinations,

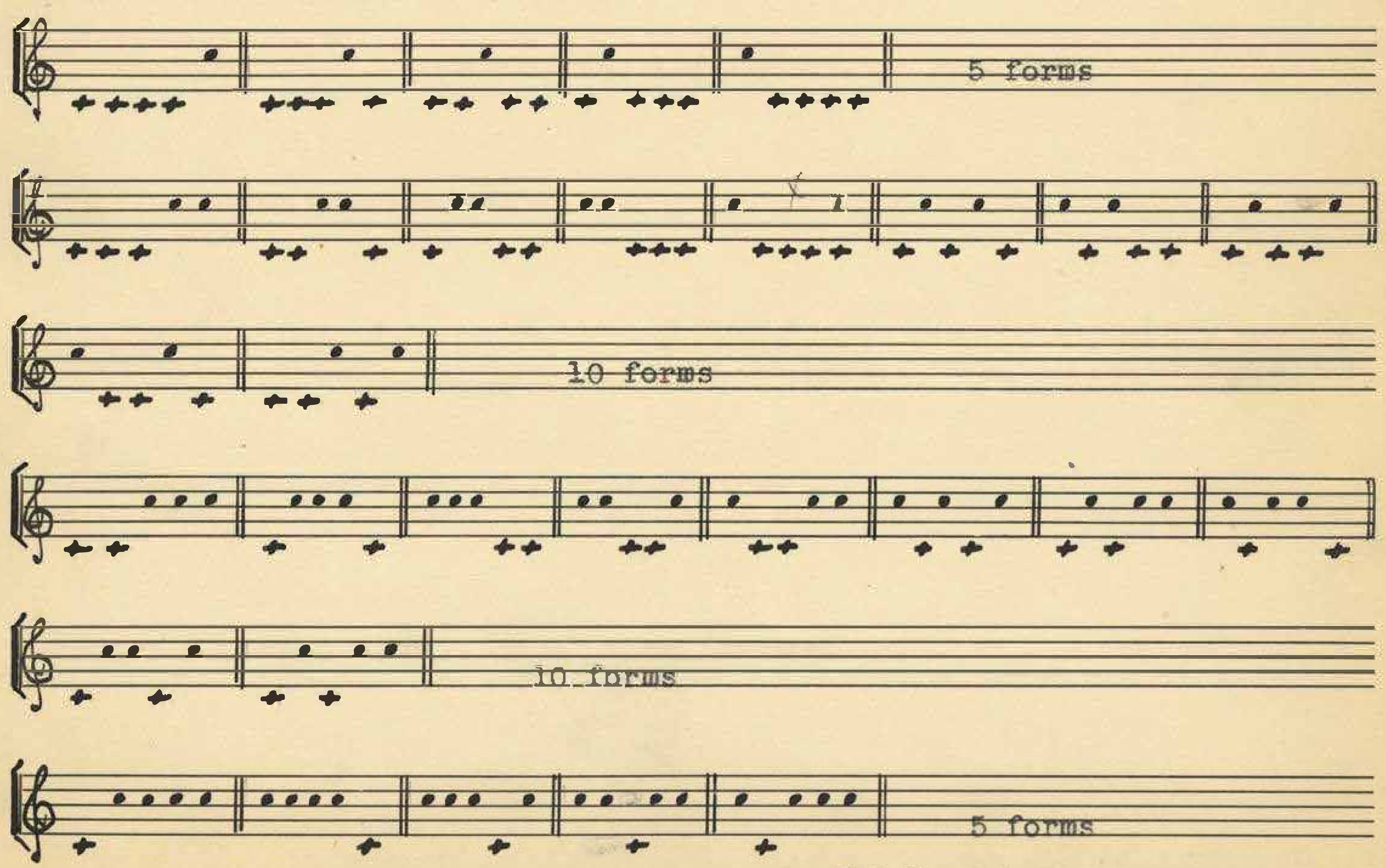
3 permutations each. Total 2·3 = 6

A = 4a: 3a+b; 2a+2b; a+3b



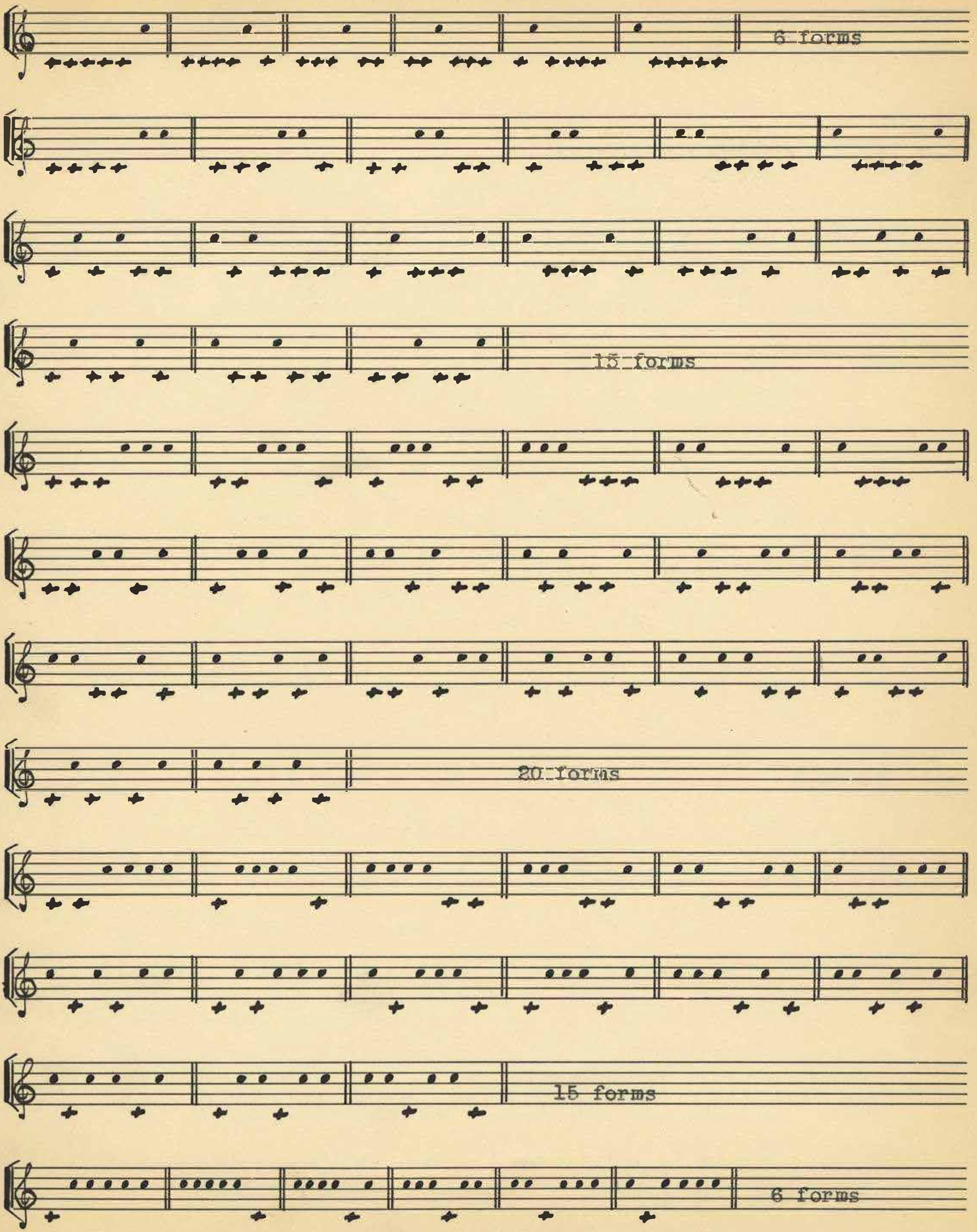
Total: 4+6+4 = 14

A = 5a: 4a+b; 3a+2b; 2a+3b; a+4b



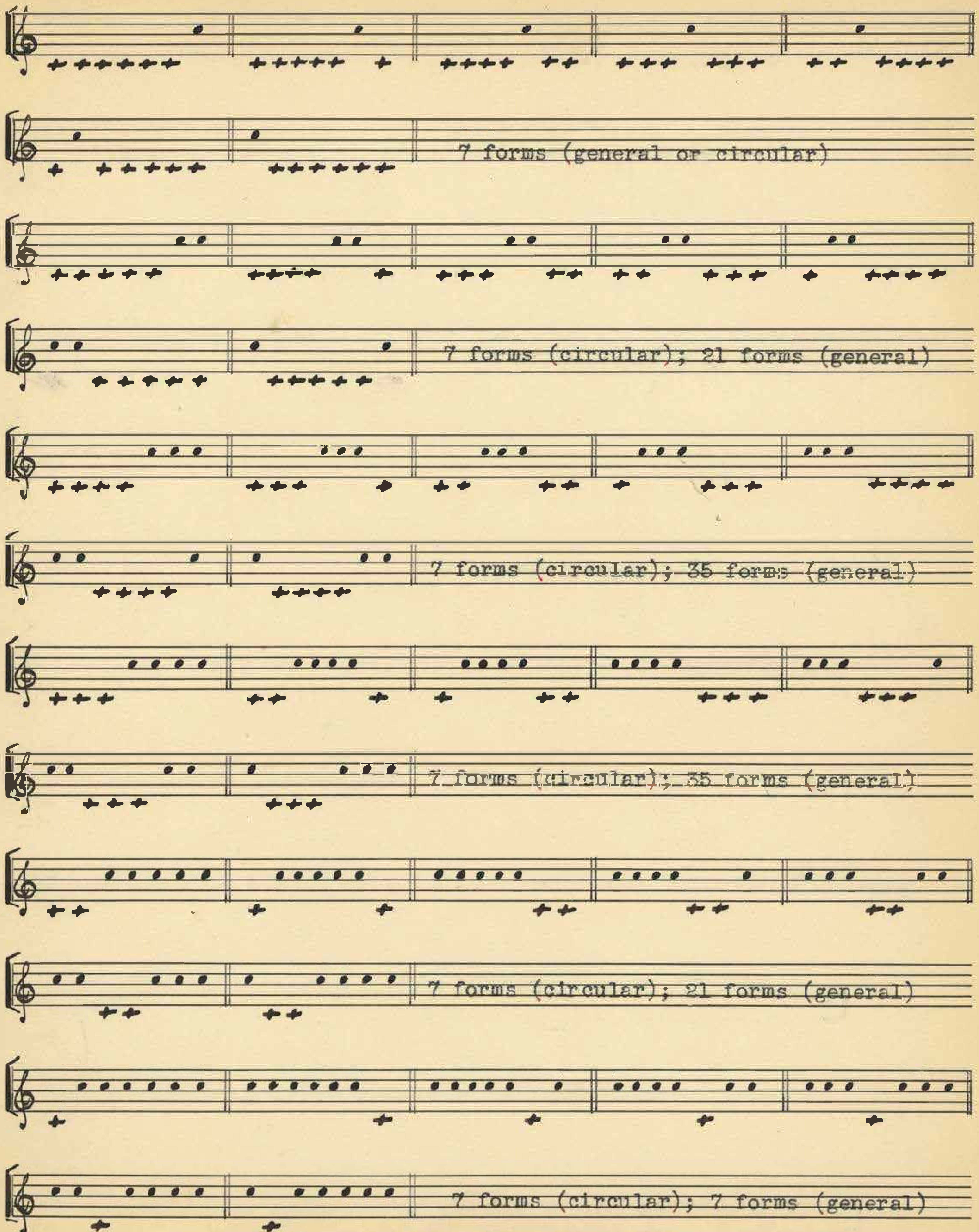
Total: 5+10+10+5 = 30

A = 6a: 5a+b; 4a+2b; 3a+3b; 2a+4b; a+5b



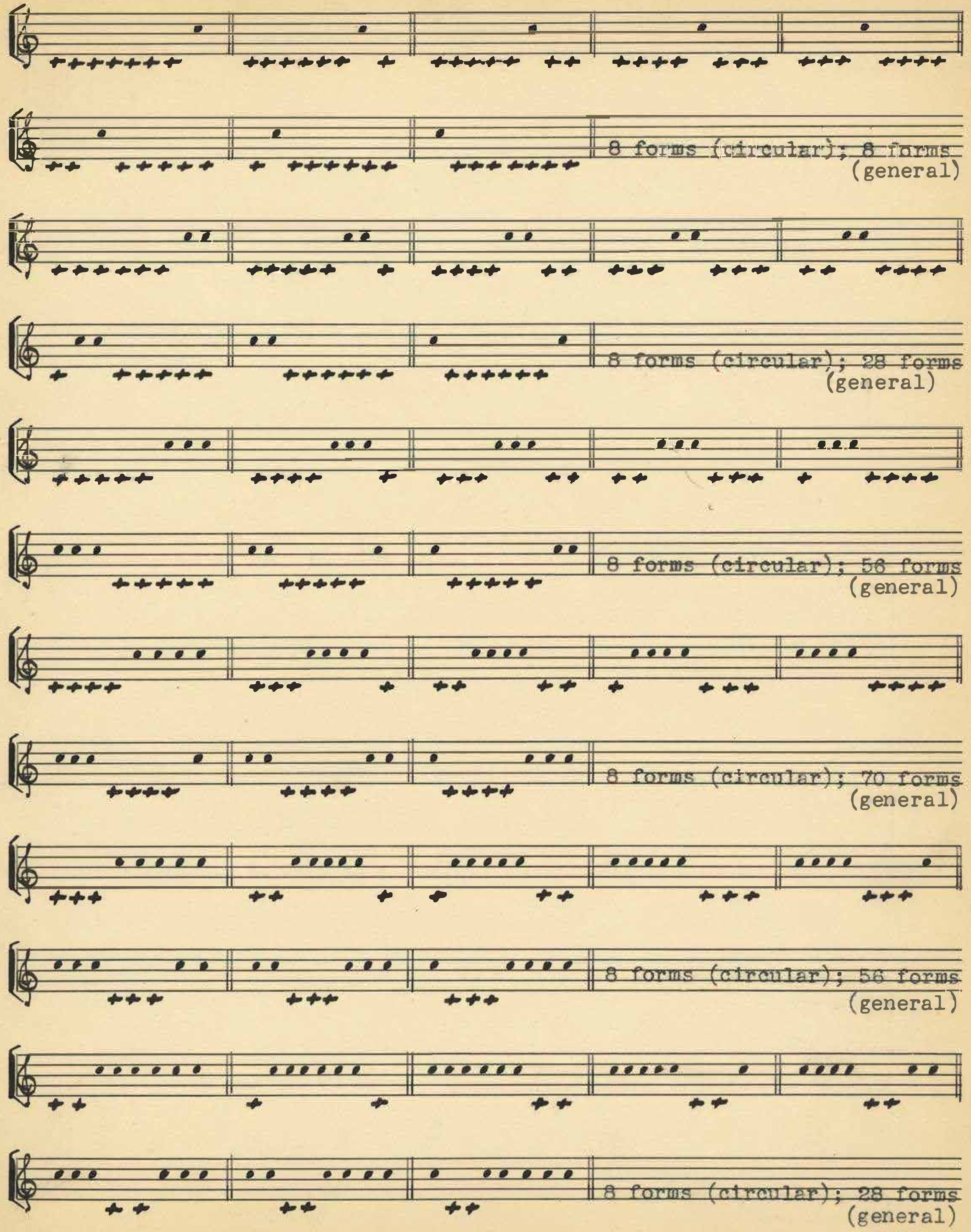
Total: 6+15+20+15+6 = 62

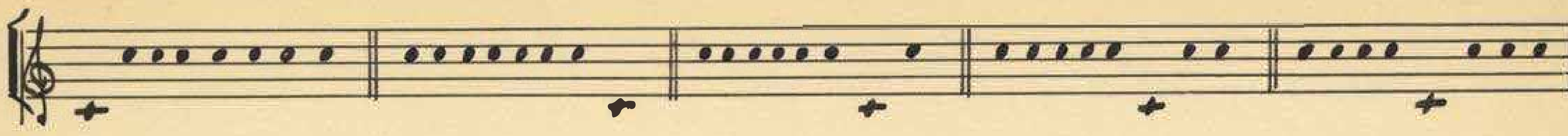
A = 7a: 6a+b; 5a+2b; 4a+3b; 3a+4b;2a+5b; a+6b



Total: 7+21+35+35+21+7 = 126

A = 8a: 7a+b; 6a+2b; 5a+3b; 4a+4b; 3a+5b; 2a+6b; a+7b

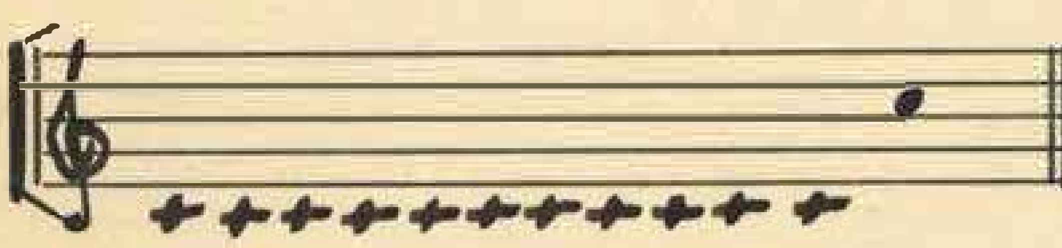




8 forms (circular); 8 forms (general)

Total: 8+28+56+70+56+28+8 = 254

A = 12a: 11a+b; 10a+2b; 9a+3b; 8a+4b; 7a+5b; 6a+6b; 5a+7b; 4a+8b; 3a+9b; 2a+10b; a+11b



12 forms (circular); 12 forms (general)

12 forms (circular); 66 forms (general)

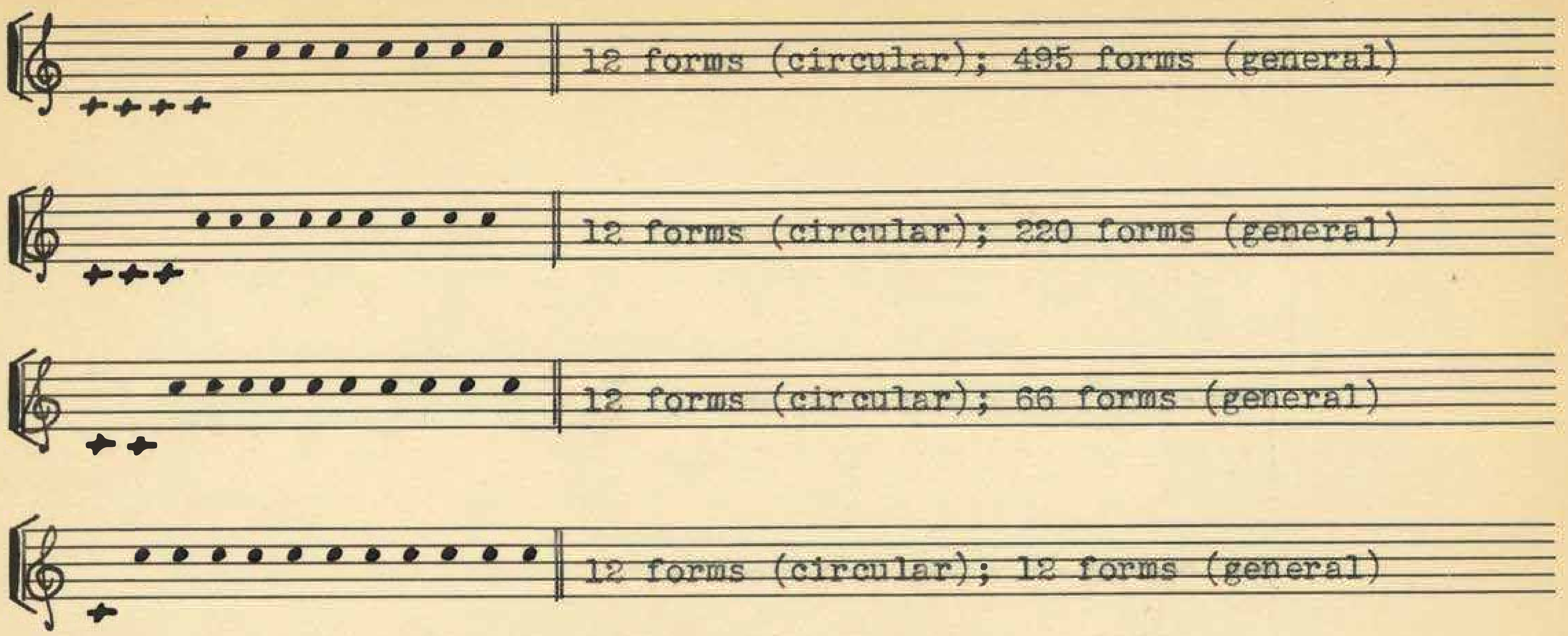
12 forms (circular); 220 forms (general)

12 forms (circular); 495 forms (general)

12 forms (circular); 792 forms (general)

12 forms (circular); 924 forms (general)

12 forms (circular); 792 forms (general)



Total: 12+66+220+495+792+924+792+495+220+66+12 = 4094

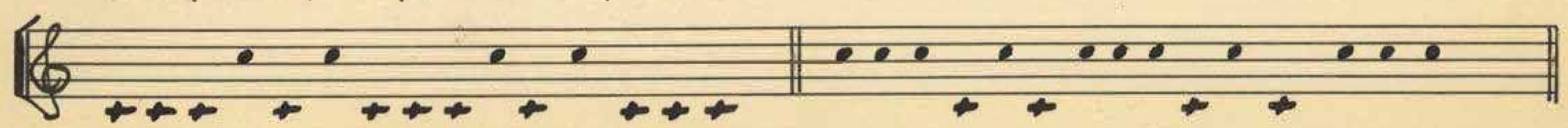
Figure III.

$A = r_{3 \div 2}$ $A = r_{4 \div 3}$ $A = r_{5 \div 4}$

A = Summation Series I A = Summation Series II

$A = (2+1+1)^2$ $A = 3(2+1) + (2+1)^2$

$A = (3+1+1) + (1+3+1) + (1+1+3)$



Lesson CLXXXIII.

Instrumental forms of $S = 2p$

Material:

(a) coupled melody: $M\left(\frac{p_{II}}{p_I}\right)$;

(b) harmonic forms of two-unit scales;

(c) two-part harmony;

(d) two-parts of any harmony.

$$I = a: \frac{p_{II}}{p_I}, \frac{p_I}{p_{II}}; \frac{a_2}{b_2} + \frac{b_2}{a_2}; \frac{ma_2}{mb_2} + \frac{nb_2}{na_2}$$

I = ab, ba: permutations of the higher orders.

Coefficients of recurrence: 2a+b; a+2b; . . . . . . ma + nb.

Figure IV.

(please see pages 24-29)

Individual attacks emphasizing one or two parts can be combined into one attack-group of any desirable form.

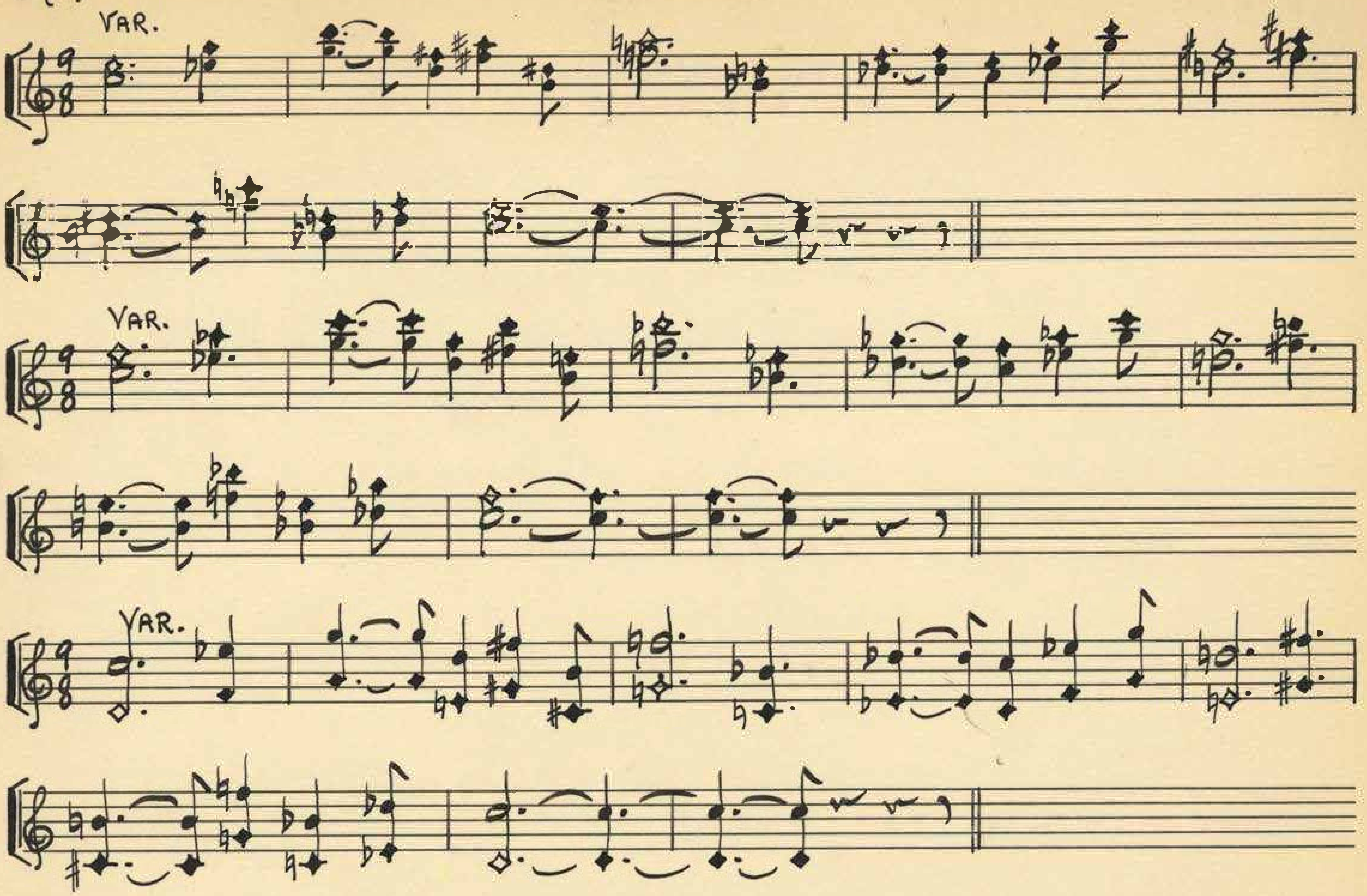
Example:

```
               b     bb     bb bbb      bbbb b
I (S = 2p):   aa ; aaaa ; aaa aa ; aa a  aa ; . . .
```

Figure V.

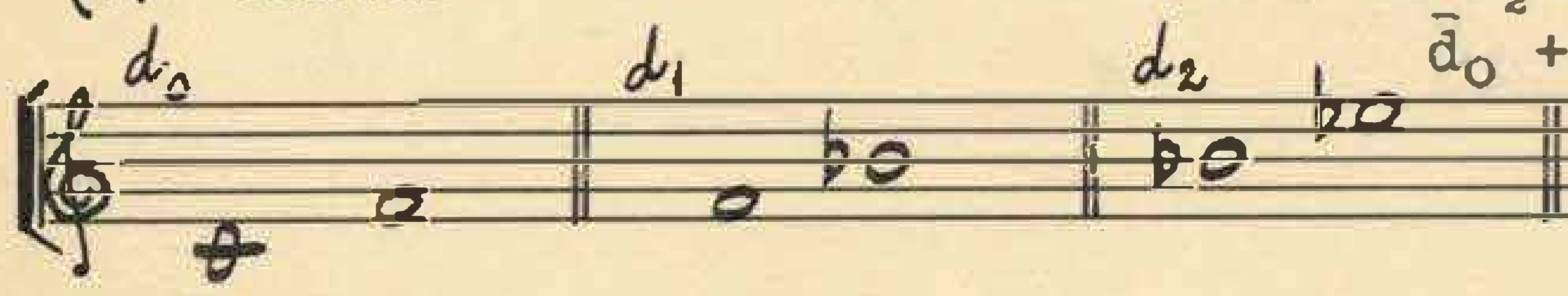
(please see page 30)

(a)



(b) SCALE

$M = 2a_2 + 2b_2 + a_2 + b_2 + 2a_2 + 2b_2$;
$\bar{d}_0 + d_2 + d_1$

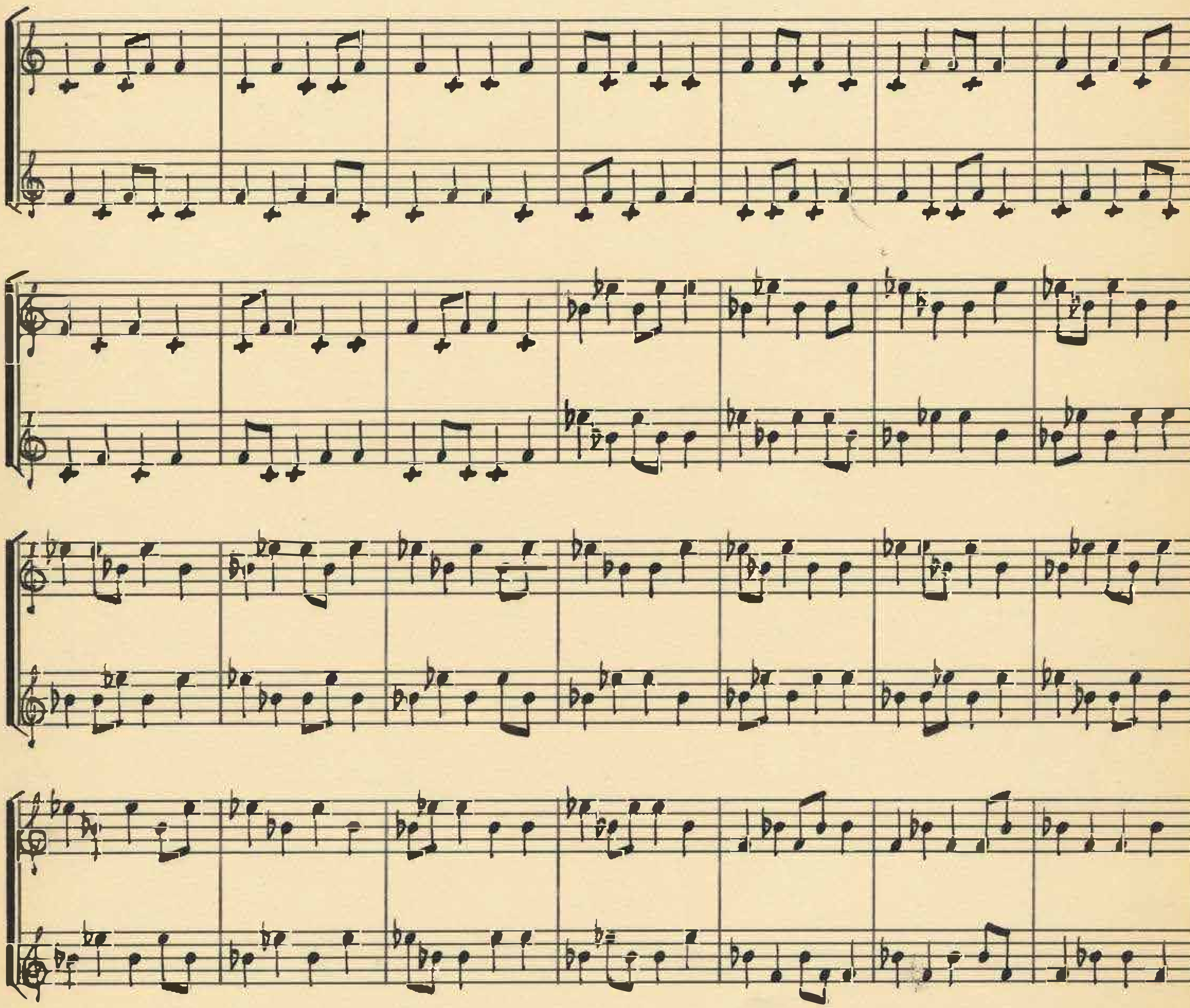


$T = r_{5 \div 2}$; $T'' = 8t$.

THEME

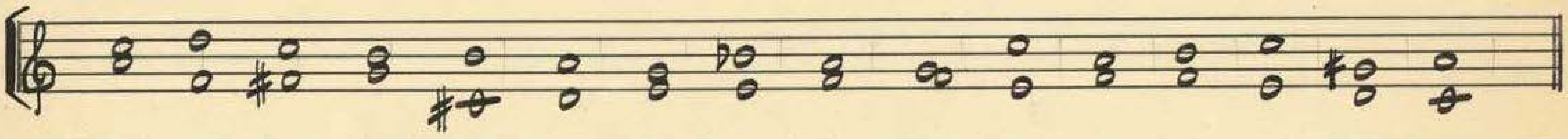
VAR I:

$$\frac{2a_2+2b_2+a_2+b_2+2a_2+2b_2}{2b_2+2a_2+b_2+a_2+2b_2+2a_2}$$



(2)

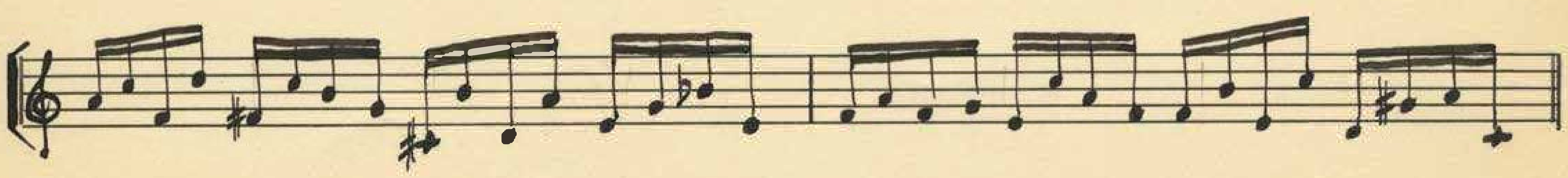
THEME: $S = 2p$



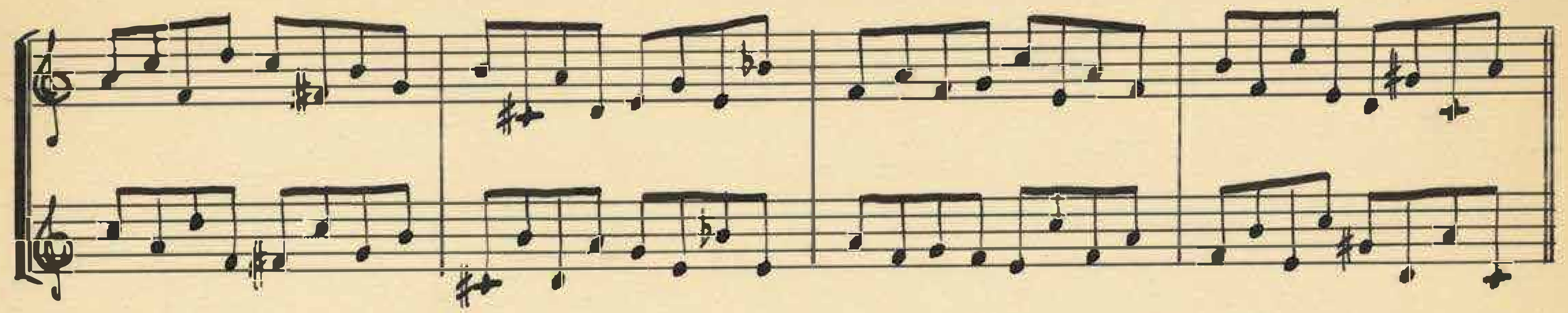
VAR. I $= 3a_2 + b_2$

VAR. I $= 3a_2 + b_2$; $T = \frac{2}{4} \div 3$; $T'' = 12t$; t = ♪

$T'' = 16t$; t = 𝅘𝅥𝅯



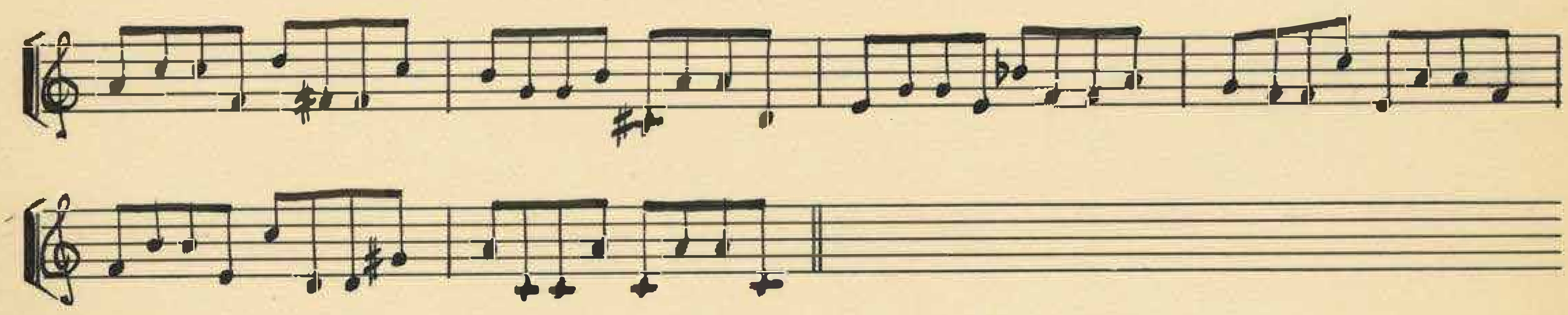
VAR. I $\frac{2a_2 + 2b_2}{2b_2 + 2a_2} + \frac{2b_2 + 2a_2}{2a_2 + 2b_2}$



When the progression of chords ($H^{-\to}$) has an assigned duration group, instrumental form (I) can be carried out through t.

THEME

VAR. I = $a_4 + b_4$; t = ♪

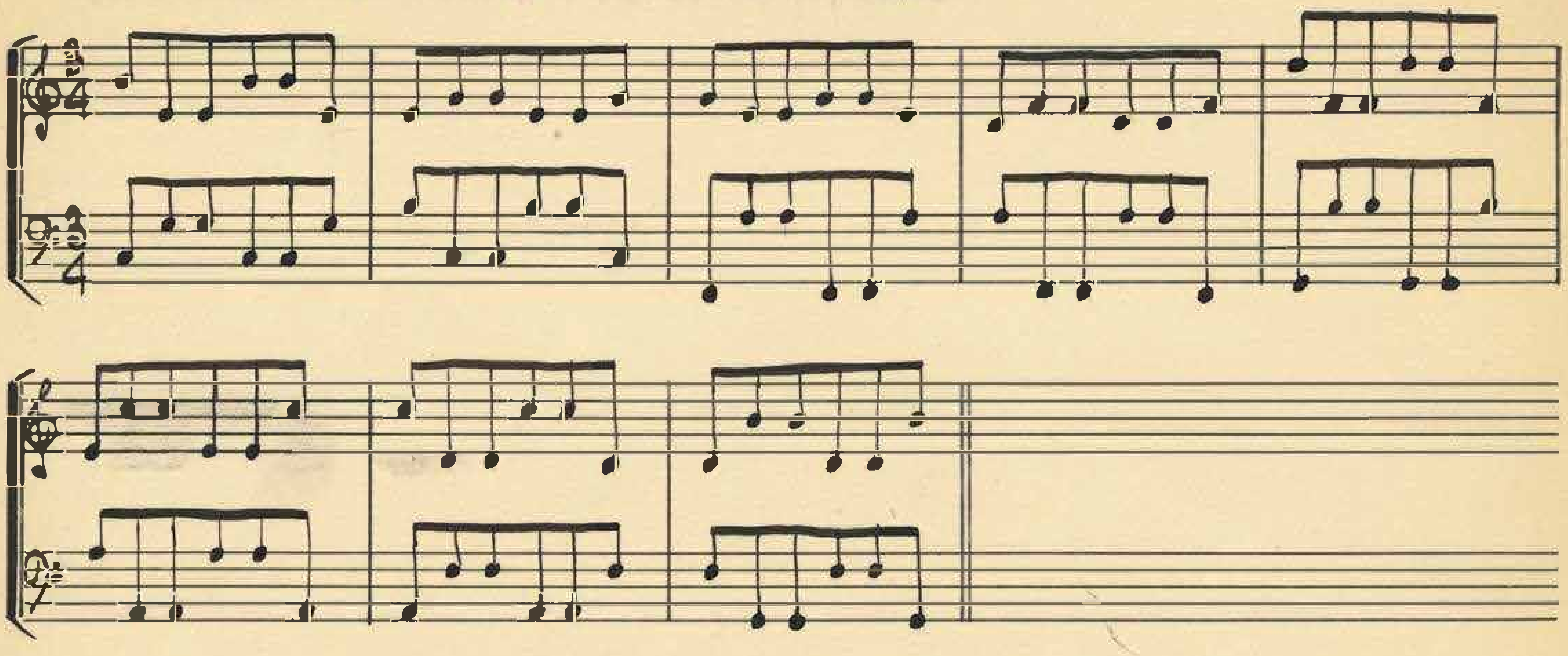


(d) THEME: S = 4 ♭

VAR. I $\left(\frac{\flat IV}{\flat III}\right) = (b_2 + a_2 + b_2)H_1 + (a_2 + b_2 + a_2)H_2$

VAR. I $\left(\frac{\flat II}{\flat I}\right) = (a_2 + b_2 + a_2)H_1 + (b_2 + a_2 + b_2)H_2$

Var.: the two preceding variations combined



THEME: $T = \tau_{5 \div 4}$

VAR. I $\left(\frac{p\,III,\ p\,IV}{p\,I,\ p\,II}\right) = \frac{2a_2 + b_2 + a_2 + 2b_2}{2b_2 + a_2 + b_2 + 2a_2}$; $t =$ ♪

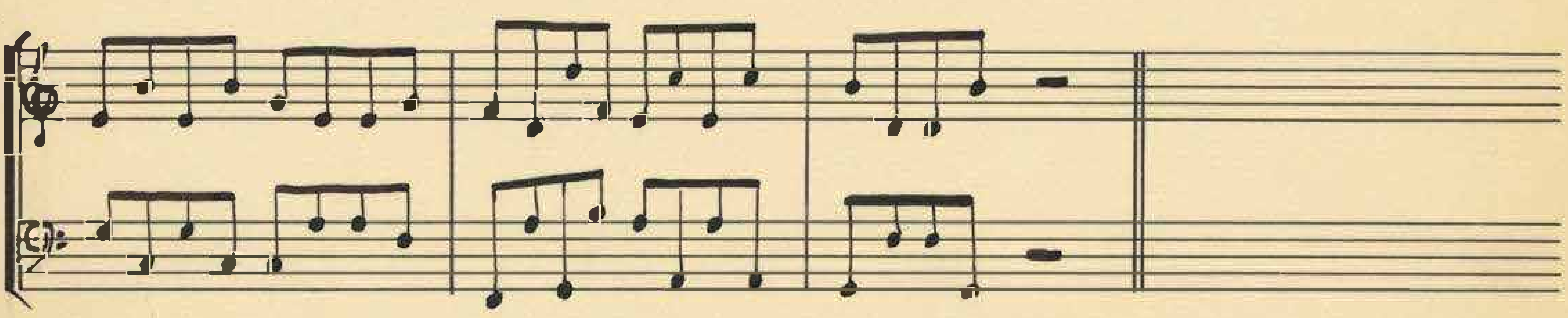
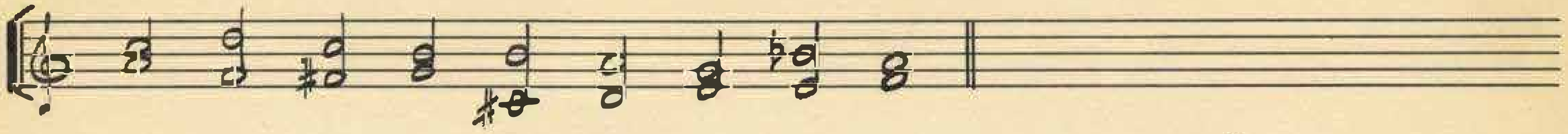


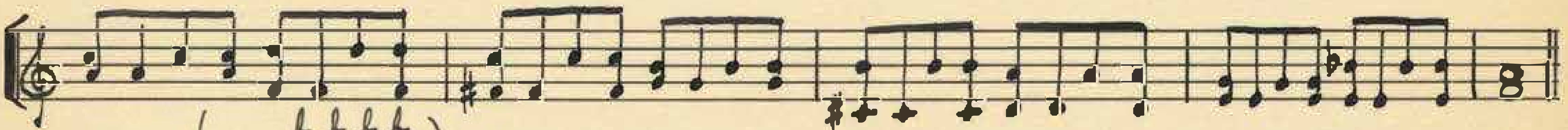
Figure V.

THEME



b bb

VAR.: aa a.



bbbb

VAR.: (aaa a) H.

bbbb

VAR.: aaa a ; H = 4a = 4 t.

Lesson CLXXXIV.

General classification of I (S = 3p)

(Table of the combinations of attacks for a, b and c)

A = a; 2a; 3a; 4a; 5a; 6a; 7a; 8a; 12a.

The following is a complete table of all forms of I(S = 3p). It includes all the combinations and permutations for 2, 3, 4, 5, 6, 7, 8 and 12 sequent attacks.

(1) I = ap (one part, one attack).

Three invariant forms: a or b or c.

A = ap, 2ap, ... map.

This is equivalent to I(S = p).

(2) I = a2p→ (one attack to a part, two sequent parts)

Three invariant forms: ab, ac, bc.

Each invariant form produces 2 attacks and has 2 permutations.

This is equivalent to I(S = 2p).

Further combinations of ab, ac, bc are not necessary as it corresponds to the forms of (3).

(3) I = a3p→ (one attack to a part, three sequent parts).

One invariant form: abc.

The invariant form produces 3 attacks and has 6 permutations:

abc, acb, cab, bac, bca, cba.

All other attack-groups (A = 3 + n) develop from this source by means of the coefficients of recurrence.

Figure VI.

I(S = 3p): attack-groups for one simultaneous p.

(please see page 33)

Development of attack-groups by means of the coefficients of recurrence.

A = 4a; 2a+b+c; a+2b+c; a+b+2c.

$$P_4 = \frac{4!}{2!} = \frac{24}{2} = 12$$

Each of the above 3 permutations of the coefficients has 12 general permutations.

Total in general permutations: 12·3 = 36

Total in circular permutations: 4·3 = 12

A = 5a.

Forms of the distribution of coefficients:

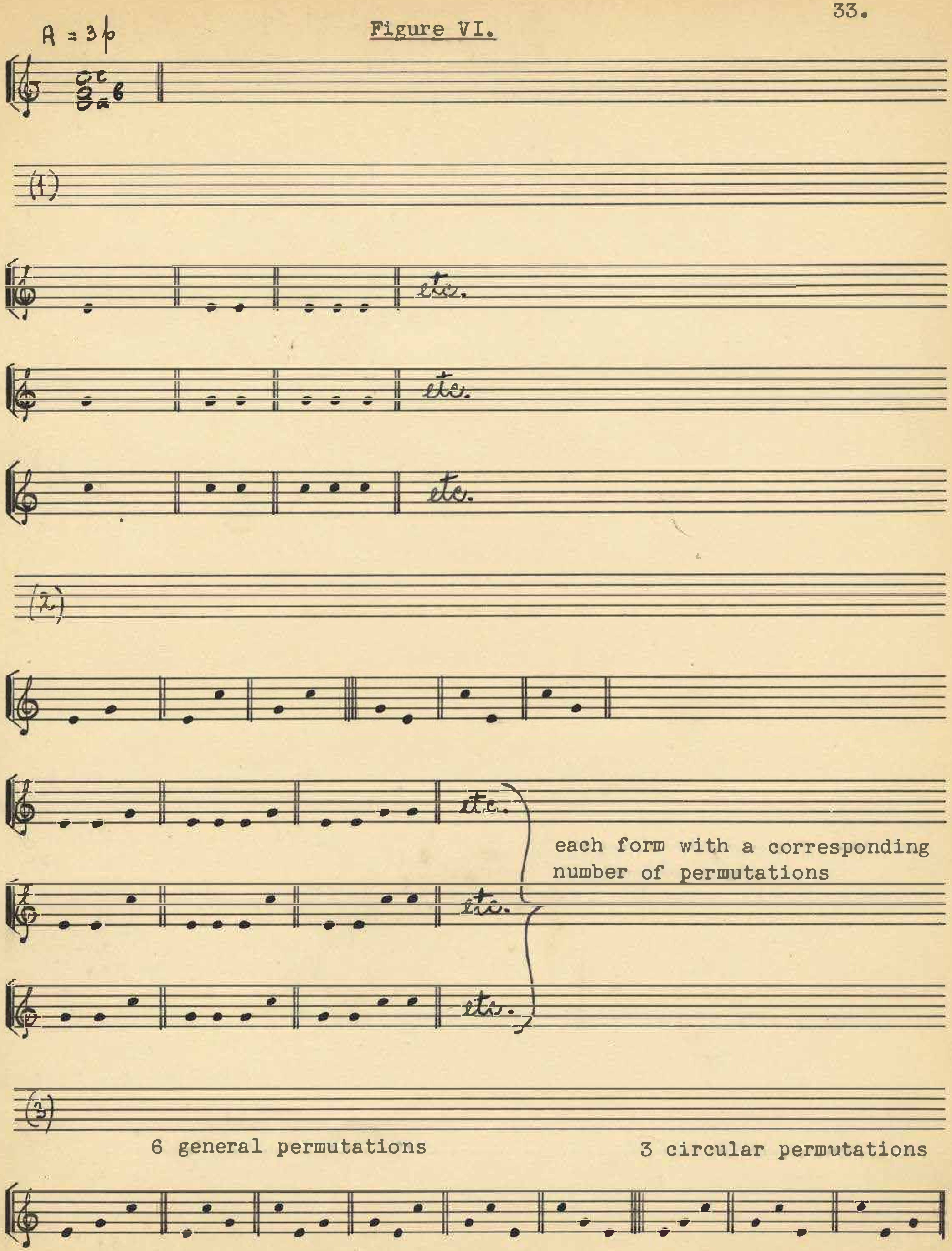
5 = 2+2+1 and 5 = 1+1+3

A = 2a+2b+c; 2a+b+2c; a+2b+2c

$$P_5 = \frac{5!}{2!\ 2!} \quad \frac{120}{2 \cdot 2} = 30$$

Each of the 3 permutations of the first form of distribution has 30 general permutations. Total: 30·3 = 90.

Figure VI.


A = 3♭
(1)
etc.
etc.
etc.
(2)
etc.
etc.
etc.
each form with a corresponding number of permutations
(3)
6 general permutations
3 circular permutations

$A = a+b+3c; \; a+3b+c; \; 3a+b+c.$

$$P_5 = \frac{5!}{3!} = \frac{120}{6} = 20$$

Each of the above 3 permutations of the second form of distribution has 20 general permutations.

Total: $20 \cdot 3 = 60$.

The total number of cases: $A = 5a$.

General permutations: $90 + 60 = 150$

Circular permutations: $5 \cdot 6 = 30$

$A = 6a$.

Forms of the distribution of coefficients:

$6 = 1+1+4; \; 1+2+3; \; 2+2+2.$

$A = a+b+4c; \; a+4b+c; \; 4a+b+c.$

$$P_6 = \frac{6!}{4!} = \frac{720}{24} = 30.$$

Each of the above 3 permutations of the first form of distribution has 30 general permutations.

Total: $30 \cdot 3 = 90$

$A = a+2b+3c; \; a+3b+2c; \; 3a+b+2c; \; 2a+b+3c;$
$2a+3b+c; \; 3a+2b+c.$

$$P_6 = \frac{6!}{2!\,3!} = \frac{720}{2 \cdot 6} = 60.$$

Each of the above 6 permutations of the second form of distribution has 60 general permutations.

Total: $60 \cdot 6 = 360$.

$A = 2a+2b+2c.$

$$P_6 = \frac{6!}{2!\ 2!\ 2!} = \frac{720}{2\cdot 2\cdot 2} = 90.$$

The third form of distribution (invariant) has 90 general permutations.

The total number of cases: $A = 6a$.

General permutations: 90 + 360 + 90 = 540.

Circular permutations: 18 + 36 + 6 = 60.

$A = 7a.$

Forms of the distribution of coefficients:

7 = 1+1+5; 1+2+4; 2+2+3; 3+3+1

$A = a+b+5c$; $a+5b+c$; $5a+b+c$.

$$P_7 = \frac{7!}{5!} = \frac{5040}{120} = 42$$

Each of the above 3 permutations of the first form of distribution has 42 general permutations.

Total: 42·3 = 126.

$A = a+2b+4c$; $a+4b+2c$; $4a+b+2c$; $2a+b+4c$; $2a+4b+c$; $4a+2b+c$.

$$P_7 = \frac{7!}{2!\ 4!} = \frac{5040}{2\cdot 24} = 105$$

Each of the above 6 permutations of the second form of distribution has 105 general permutations.

Total: 105·6 = 630

$A = 2a+2b+3c$; $2a+3b+2c$; $3a+2b+2c$.

$$P_7 = \frac{7!}{2!\ 3!\ 2!} = \frac{5040}{2\cdot 6\cdot 2} = 210$$

Each of the above 3 permutations of the third form of distribution has 210 general permutations.

Total: 210·3 = 630

A = 3a+3b+c; 3a+b+3c; a+3b+3c.

$$P_7 = \frac{7!}{3!\ 3!} = \frac{5040}{6 \cdot 6} = 140$$

Each of the above 3 permutations of the fourth form of distribution has 140 general permutations.

Total: 140·3 = 420

The total number of cases: A = 7a

General permutations: 126 + 630 + 630 + 420 = 1806

Circular permutations: 21 + 42 + 21 + 21 = 105

A = 8a.

Forms of the distribution of coefficients:

8 = 1+1+6; 1+2+5; 1+3+4; 2+2+4; 2+3+3

A = a+b+6c; a+6b+c; 6a+b+c.

$$P_8 = \frac{8!}{6!} = \frac{40{,}320}{720} = 56$$

Each of the above 3 permutations of the first form of distribution has 56 general permutations.

Total: 56·3 = 168

A = a+2b+5c; a+5b+2c; 5a+b+2c;

2a+b+5c; 2a+5b+c; 5a+2b+c.

$$P_8 = \frac{8!}{2!\ 5!} = \frac{40{,}320}{2 \cdot 120} = 168$$

Each of the above 6 permutations of the second form of distribution has 168 general permutations.

Total: 168·6 = 1008

$A = a+3b+4c; \quad a+4b+3c; \quad 4a+b+3c;$

$3a+b+4c; \quad 3a+4b+c; \quad 4a+3b+c.$

$$P_8 = \frac{8!}{3!\,4!} = \frac{40,320}{6\cdot 24} = 280$$

Each of the above 6 permutations of the third form of distribution has 280 general permutations.

Total: $280\cdot 6 = 1680$

$A = 2a+2b+4c; \quad 2a+4b+2c; \quad 4a+2b+2c$

$$P_8 = \frac{8!}{2!\,2!\,4!} = \frac{40,320}{2\cdot 2\cdot 24} = 420$$

Each of the above 3 permutations of the fourth form of distribution has 420 general permutations.

Total: $420\cdot 3 = 1260$

$A = 2a+3b+3c; \quad 3a+2b+3c; \quad 3a+3b+2c$

$$P_8 = \frac{8!}{2!\,3!\,3!} = \frac{40,320}{2\cdot 6\cdot 6} = 560$$

Each of the above 3 permutations of the fifth form of distribution has 560 general permutations.

Total: $560\cdot 3 = 1680$

The total number of cases: $A = 8a$

General permutations: $168 + 1008 + 1680 + 1260 +$
$+ 1680 = 5796$

Circular permutations: $24 + 48 + 48 + 24 + 24 = 168$

$A = 12a.$

Forms of the distribution of coefficients:

$8 = 1+1+10; \; 1+2+9; \; 1+3+8; \; 1+4+7; \; 1+5+6; \; 2+2+8;$
$2+3+7; \; 2+4+6; \; 2+5+5; \; 3+3+6; \; 3+4+5; \; 4+4+4.$

A = a+b+10c; a+10b+c; 10a+b+c

$$P_{12} = \frac{12!}{10!} = \frac{479,001,600}{3,628,800} = 132$$

Each of the above 3 permutations of the first form of distribution has 132 general permutations.

Total: 132·3 = 396

A = a+2b+9c; a+9b+2c; 9a+b+2c;
2a+b+9c; 2a+9b+c; 9a+2b+c.

$$P_{12} = \frac{12!}{2!\ 9!} = \frac{479,001,600}{2 \cdot 362,880} = 660$$

Each of the above 6 permutations of the second form of distribution has 660 general permutations.

Total: 660·6 = 3960

A = a+3b+8c; a+8b+3c; 8a+b+3c;
3a+b+8c; 3a+8b+c; 8a+3b+c.

$$P_{12} = \frac{12!}{3!\ 8!} = \frac{479,001,600}{6 \cdot 40,320} = 1980$$

Each of the above 6 permutations of the third form of distribution has 1980 general permutations.

Total: 1980·6 = 11,880

A = a+4b+7c; a+7b+4c; 7a+b+4c;
4a+b+7c; 4a+7b+c; 7a+4b+c.

$$P_{12} = \frac{12!}{4!\ 7!} = \frac{479,001,600}{24 \cdot 5,040} = 3960$$

Each of the above 6 permutations of the fourth form of distribution has 3960 general permutations.

Total: 3960·6 = 23,760

A = a+5b+6c; a+6b+5a; 6a+b+5c;

5a+b+6c; 5a+6b+c; 6a+5b+c.

$$P_{12} = \frac{12!}{5!\ 6!} = \frac{479,001,600}{120\cdot720} = 5544$$

Each of the above 6 permutations of the fifth form of distribution has 5544 general permutations.

Total: 5544·6 = 32,264

A = 2a+2b+8c; 2a+8b+2c; 8a+2b+2c

$$P_{12} = \frac{12!}{2!\ 2!\ 8!} = \frac{479,001,600}{2\cdot2\cdot40,320} = 2970$$

Each of the above 3 permutations of the sixth form of distribution has 2970 general permutations.

Total: 2970·3 = 8910

A = 2a+3b+7c; 2a+7b+3c; 7a+2b+3c;

3a+2b+7c; 3a+7b+2c; 7a+3b+2c.

$$P_{12} = \frac{12!}{2!\ 3!\ 7!} = \frac{479,001,600}{2\cdot6\cdot5,040} = 7920$$

Each of the above 6 permutations of the seventh form of distribution has 7920 general permutations.

Total: 7920·6 = 47,520

A = 2a+4b+6c; 2a+6b+4c; 6a+2b+4c;

4a+2b+6c; 4a+6b+2c; 6a+4b+2c.

$$P_{12} = \frac{12!}{2!\ 4!\ 6!} = \frac{479,001,600}{2\cdot24\cdot720} = 1386$$

Each of the above 6 permutations of the eighth form of distribution has 1386 general permutations.

Total: 1386·6 = 8316

$A = 2a+5b+5c; \quad 5a+2b+5c; \quad 5a+5b+2c.$

$$P_{12} = \frac{12!}{2!\ 5!\ 5!} = \frac{479,001,600}{2\cdot 120\cdot 120} = 16,632$$

Each of the above 3 permutations of the ninth form of distribution has 16,632 general permutations.

Total: $16,632\cdot 3 = 49,896$

$A = 3a+3b+6c; \quad 3a+6b+3c; \quad 6a+3b+3c$

$$P_{12} = \frac{12!}{3!\ 3!\ 6!} = \frac{479,001,600}{6\cdot 6\cdot 720} = 18,480$$

Each of the above 3 permutations of the tenth form of distribution has 18,480 general permutations.

Total: $18,480\cdot 3 = 55,440$

$A = 3a+4b+5c; \quad 3a+5b+4c; \quad 5a+3b+4c;$

$4a+3b+5c; \quad 4a+5b+3c; \quad 5a+4b+3c.$

$$P_{12} = \frac{12!}{3!\ 4!\ 5!} = \frac{479,001,600}{6\cdot 24\cdot 120} = 27,720$$

Each of the above 6 permutations of the eleventh form of distribution has 27,720 general permutations.

Total: $27,720\cdot 6 = 166,320$

$A = 4a+4b+4c$

$$P_{12} = \frac{12!}{4!\ 4!\ 4!} = \frac{479,001,600}{24\cdot 24\cdot 24} = 34,650$$

The twelfth form of distribution (invariant) has 34,650 general permutations.

The total number of cases: $A = 12a$.

General permutations: 396 + 3960 + 11,880 + 23,760 + + 32,264 + 8910 + 47,520 + 8316 + 49,896 + 55,440 + 166,320 + + 34,650 = 443,312.

Circular permutations: 36 + 72 + 72 + 72 + 72 + 36 + + 72 + 72 + 36 + 36 + 72 + 12 = 660.

Figure VII.

A = 4a; 2a+b+c; a+2b+c; a+b+2c

12 general or 4 circular permutations

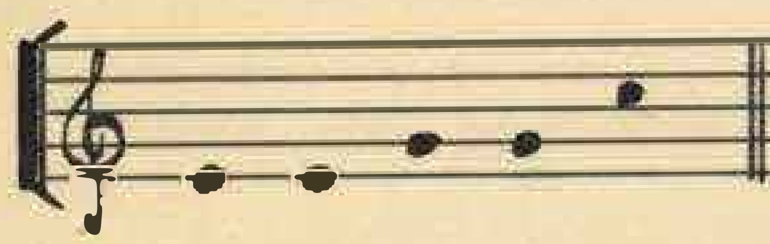
12 general or 4 circular permutations

12 general or 4 circular permutations

Total in general permutations: 12+12+12 = 36

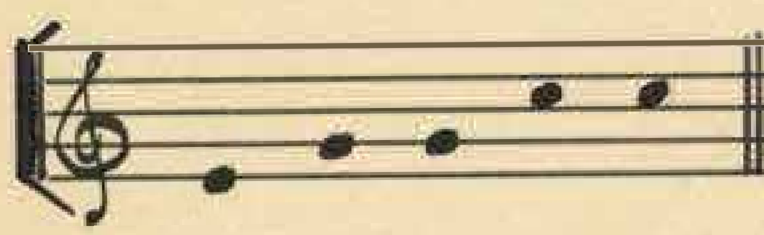
Total in circular permutations: 4+4+4 = 12

A = 5a; 2a+2b+c; 2a+b+2c; a+2b+2c



30 general or 5 circular permutations

30 general or 5 circular permutations

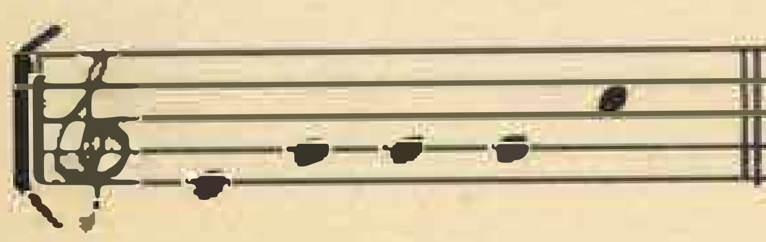


30 general or 5 circular permutations

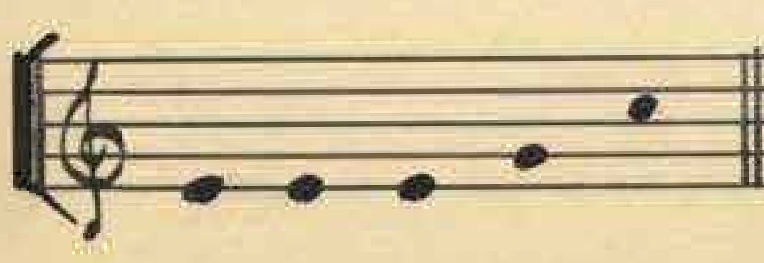
Total in general permutations: 30+30+30 = 90
Total in circular permutations: 5+5+5 = 15

A = 5a; a+b+3c; a+3b+c; 3a+b+c

20 general or 5 circular permutations



20 general or 5 circular permutations



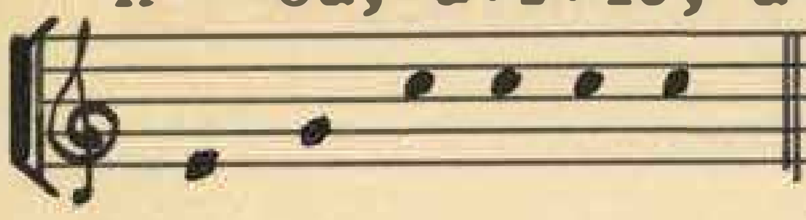
20 general or 5 circular permutations

Total in general permutations: 20+20+20 = 60
Total in circular permutations: 5+5+5 = 15

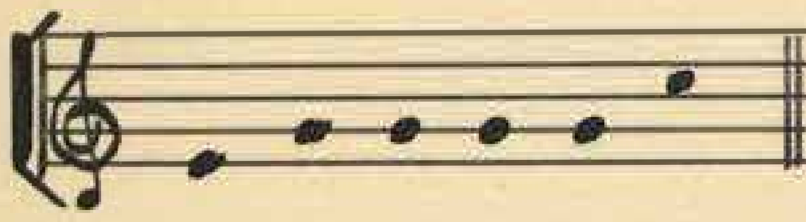
The entire total for 5 attacks: in general permutations: 150

in circular permutations: 30

A = 6a; a+b+4c; a+4b+c; a+b+4c



30 general or 6 circular permutations



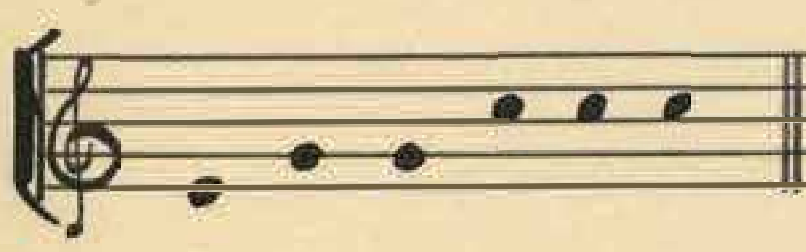
30 general or 6 circular permutations

30 general or 6 circular permutations

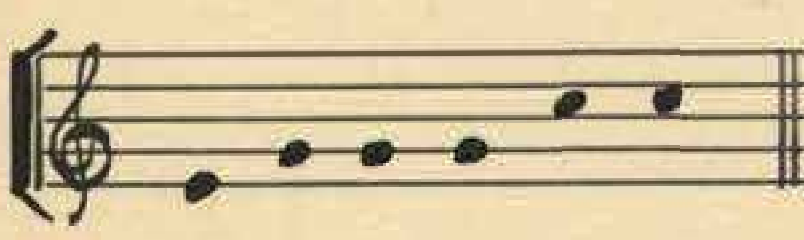
Total in general permutations: 30•3 = 90

Total in circular permutations: 6•3 = 18

A = 6a; a+2b+3c; a+3b+2c; 3a+b+2c; 2a+b+3c; 2a+3b+c; 3a+2b+c



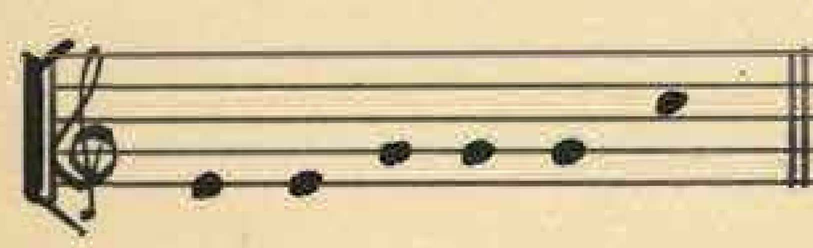
60 general or 6 circular permutations



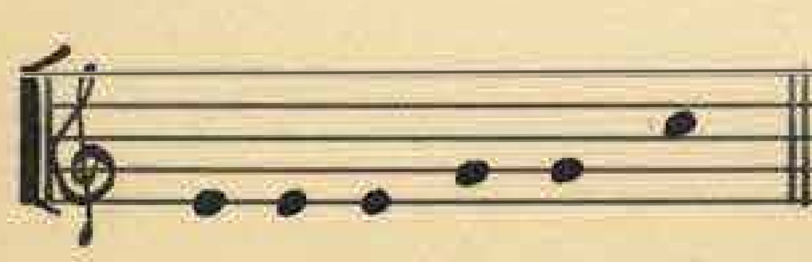
60 general or 6 circular permutations

60 general or 6 circular permutations

60 general or 6 circular permutations



60 general or 6 circular permutations



60 general or 6 circular permutations

Total in general permutations: 60•6 = 360

Total in circular permutations: 6•6 = 36

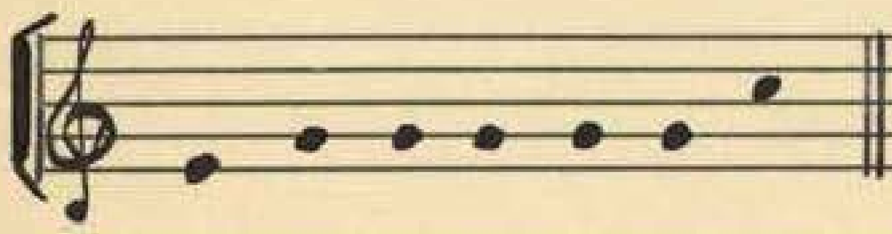
A = 6a; 2a+2b+2c

90 general or 6 circular permutations

The entire total for 6 attacks: in general permutations: 540
in circular permutations: 60

A = 7a; a+b+5c; a+5b+c; 5a+b+c

42 general or 7 circular permutations



42 general or 7 circular permutations

42 general or 7 circular permutations

Total in general permutations: 42•3 = 126
Total in circular permutations: 7•3 = 21

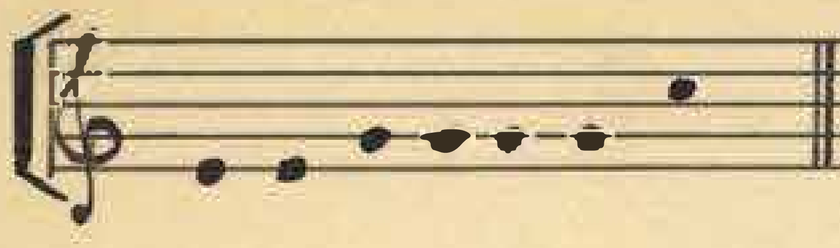
A = 7a; a+2b+4c; a+4b+2c; 4a+b+2c; 2a+b+4c; 2a+4b+c; 4a+2b+c

105 general or 7 circular permutations

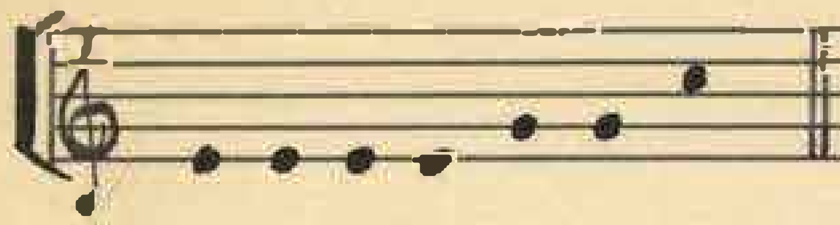
105 general or 7 circular permutations

105 general or 7 circular permutations

105 general or 7 circular permutations



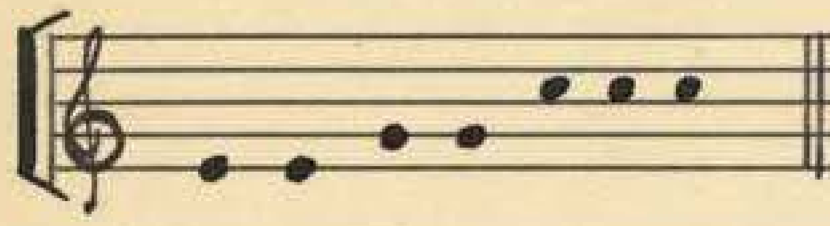
105 general or 7 circular permutations



105 general or 7 circular permutations

Total in general permutations: 105•6 = 630
Total in circular permutations: 7•6 = 42

A = 7a; 2a+2b+3c; 2a+3b+2c; 3a+2b+2c



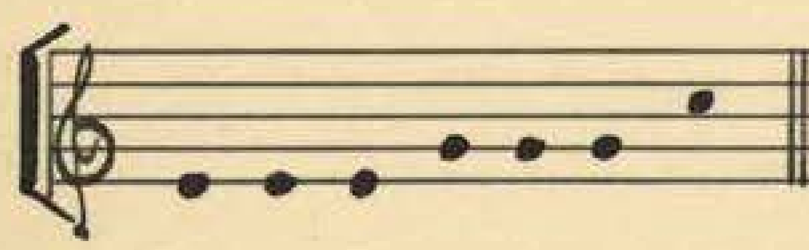
210 general or 7 circular permutations

210 general or 7 circular permutations

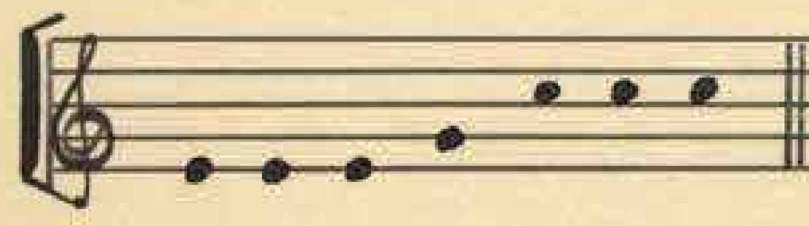
210 general or 7 circular permutations

Total in general permutations: 210•3 = 630
Total in circular permutations: 7•3 = 21

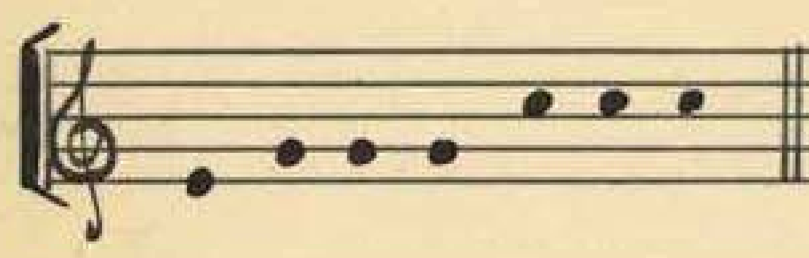
A = 7a; 3a+3b+c; 3a+b+3c; a+3b+3c



140 general or 7 circular permutations



140 general or 7 circular permutations

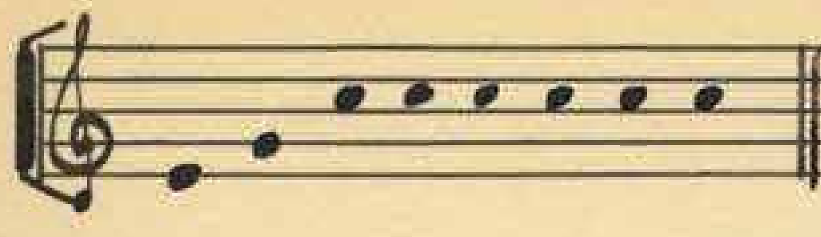


140 general or 7 circular permutations

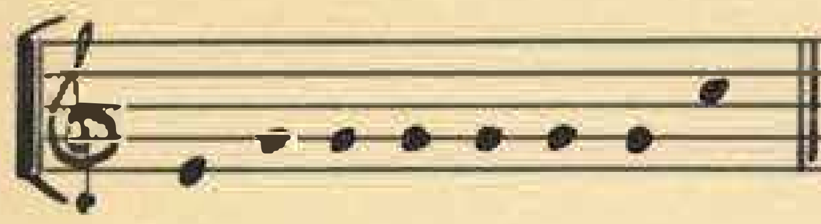
Total in general permutations: 140•3 = 420
Total in circular permutations: 7•3 = 21

The entire total for 7 attacks: in general permutations: 1806
in circular permutations: 105

A = 8a; a+b+6c; a+6b+c; 6a+b+c



56 general or 8 circular permutations



56 general or 8 circular permutations

56 general or 8 circular permutations

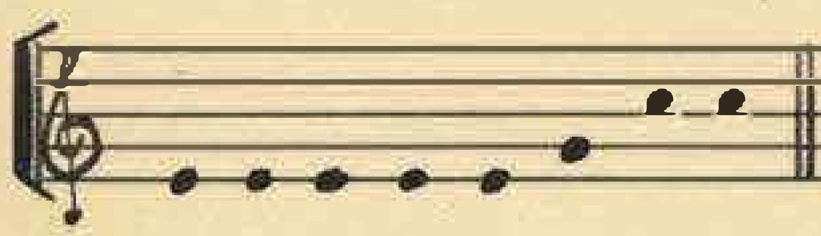
Total in general permutations: 56·3 = 168

Total in circular permutations: 8·3 = 24

A = 8a; a+2b+5c; a+5b+2c; 5a+b+2c; 2a+b+5c; 2a+5b+c; 5a+2b+c

168 general or 8 circular permutations

168 general or 8 circular permutations



168 general or 8 circular permutations

168 general or 8 circular permutations

168 general or 8 circular permutations

168 general or 8 circular permutations

Total in general permutations: 168·6 = 1008

Total in circular permutations: 8·6 = 48

A = 8a; a+3b+4c; a+4b+3c; 4a+b+3c;
3a+b+4c; 3a+4b+c; 4a+3b+c

280 general or 8 circular permutations

280 general or 8 circular permutations

280 general or 8 circular permutations

280 general or 8 circular permutations

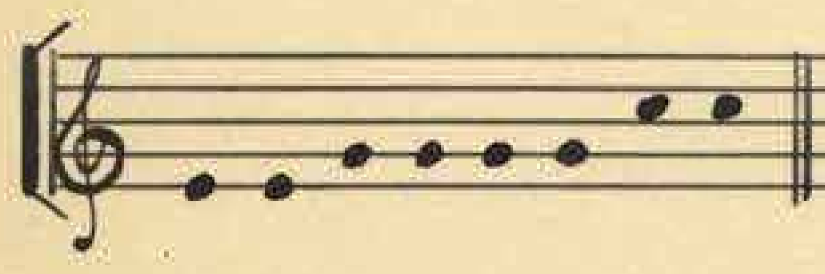
280 general or 8 circular permutations

280 general or 8 circular permutations

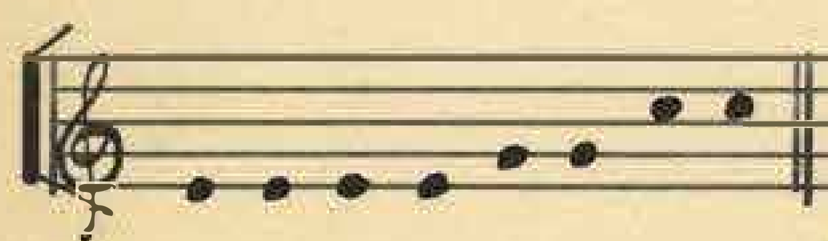
Total in general permutations: 280·6 = 1680
Total in circular permutations: 8·6 = 48

A = 8a; 2a+2b+4c; 2a+4b+2c; 4a+2b+2c

420 general or 8 circular permutations



420 general or 8 circular permutations



420 general or 8 circular permutations

Total in general permutations: 420·3 = 1260
Total in circular permutations: 8·3 = 24

A = 8a; 2a+3b+3c; 3a+2b+3c; 3a+3b+2c

560 general or 8 circular permutations

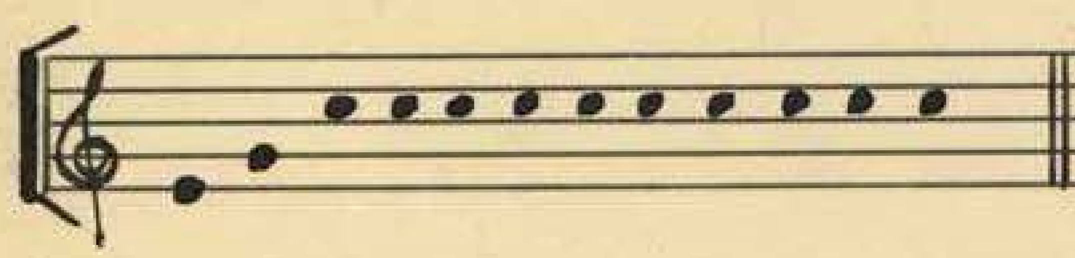
560 general or 8 circular permutations

560 general or 8 circular permutations

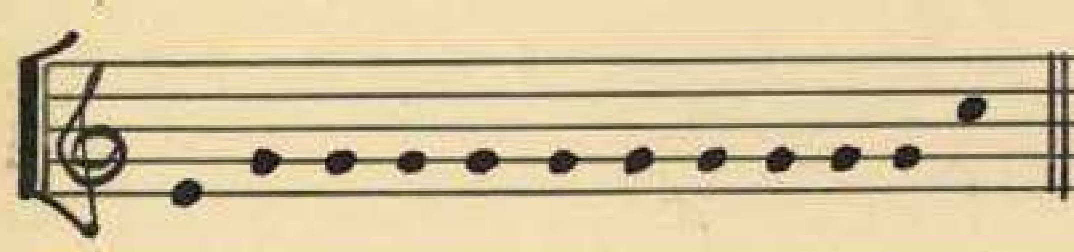
Total in general permutations: 560·3 = 1680
Total in circular permutations: 8·3 = 24

The entire total for 8 attacks: in general permutations: 5796
in circular permutations: 168

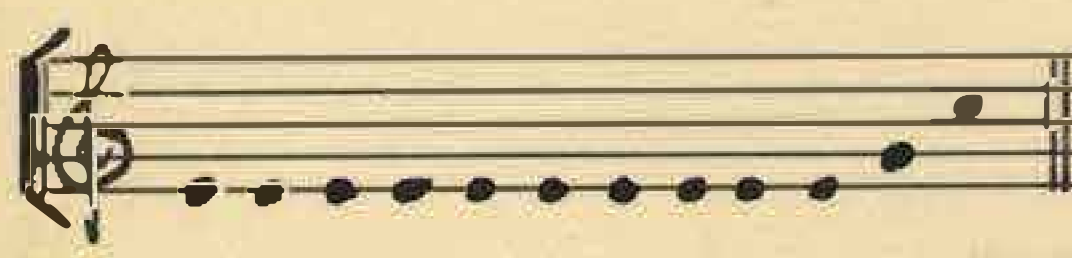
A = 12a; a+b+10c; a+10b+c; 10a+b+c



132 general or 12 circular permutations



132 general or 12 circular permutations



132 general or 12 circular permutations

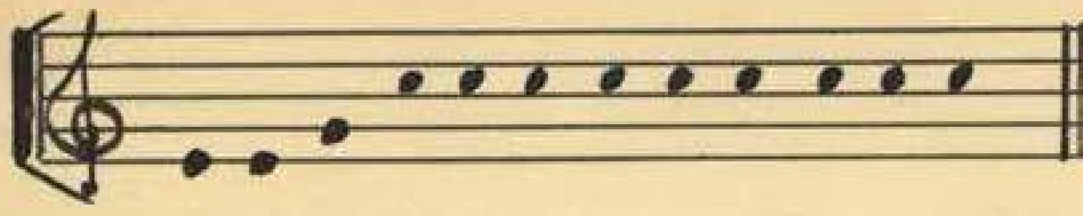
Total in general permutations: 132·3 = 396
Total in circular permutations: 12·3 = 36

A = 12a; a+2b+9c; a+9b+2c; 9a+b+2c;
2a+b+9c; 2a+9b+c; 9a+2b+c

660 general or 12 circular permutations

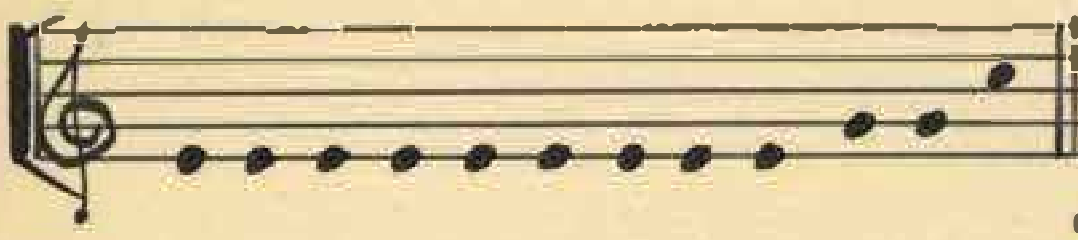
660 general or 12 circular permutations

660 general or 12 circular permutations



660 general or 12 circular permutations

660 general or 12 circular permutations

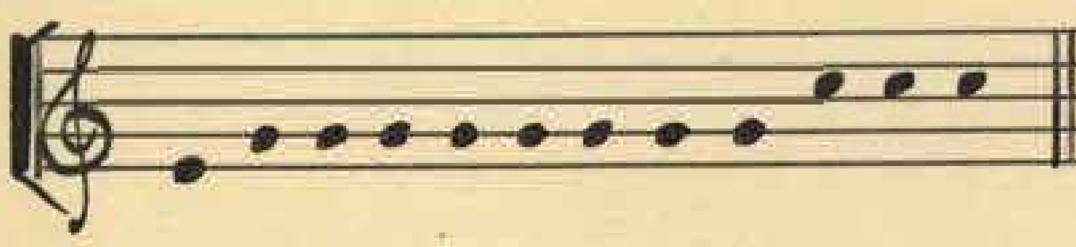


660 general or 12 circular permutations

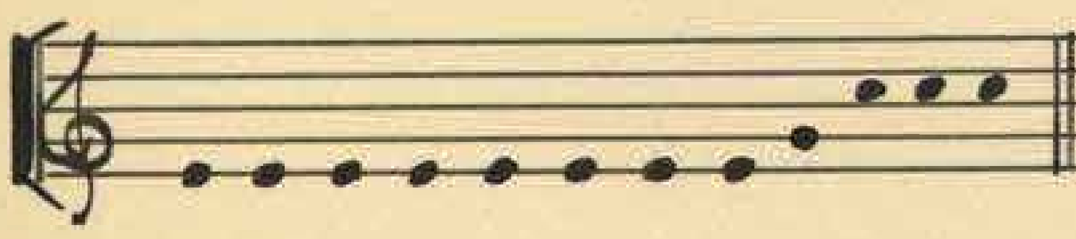
Total in general permutations: 660·6 = 3960
Total in circular permutations: 12·6 = 72

A = 12a; a+3b+8c; a+8b+3c; 8a+b+3c;
3a+b+8c; 3a+8b+c; 8a+3b+c

1980 general or 12 circular permutations



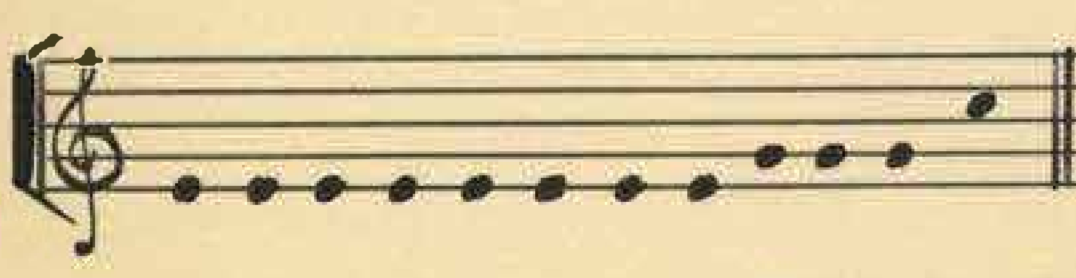
1980 general or 12 circular permutations



1980 general or 12 circular permutations

1980 general or 12 circular permutations

1980 general or 12 circular permutations



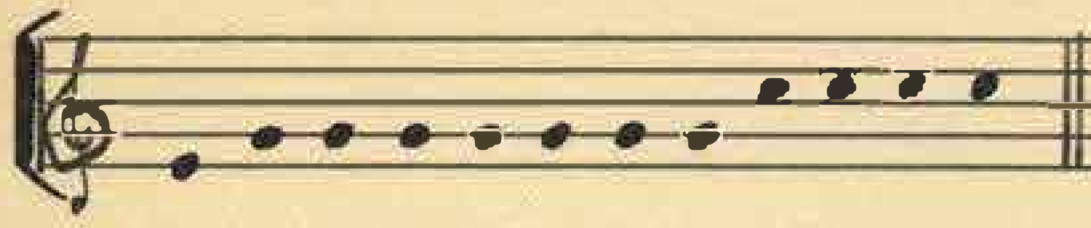
1980 general or 12 circular permutations

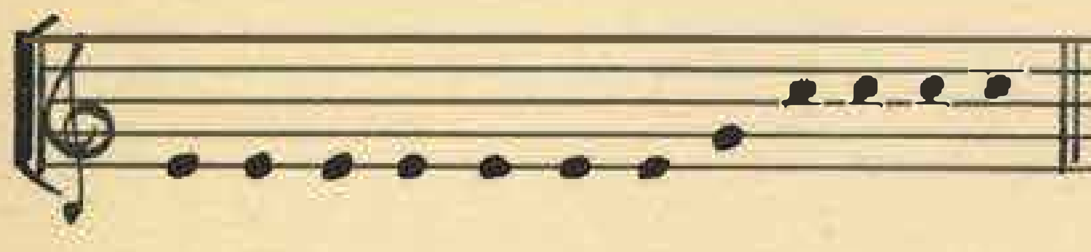
Total in general permutations: 1980·6 = 11,880

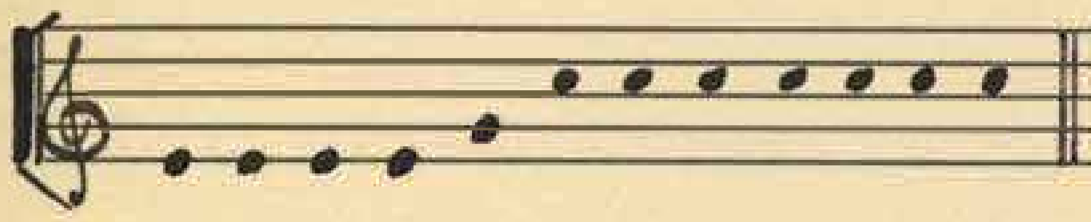
Total in circular permutations: 12·6 = 72

A = 12a; a+4b+7c; a+7b+4c; 7a+b+4c;
4a+b+7c; 4a+7b+c; 7a+4b+c


3960 general or 12 circular permutations


3960 general or 12 circular permutations


3960 general or 12 circular permutations


3960 general or 12 circular permutations


3960 general or 12 circular permutations


3960 general or 12 circular permutations

Total in general permutations: 3960.6 = 23,760
Total in circular permutations: 12.6 = 72

A = 12a; a+5b+6c; a+6b+5c; 6a+b+5c;
5a+b+6c; 5a+6b+c; 6a+5b+c


5544 general or 12 circular permutations


5544 general or 12 circular permutations


5544 general or 12 circular permutations


5544 general or 12 circular permutations

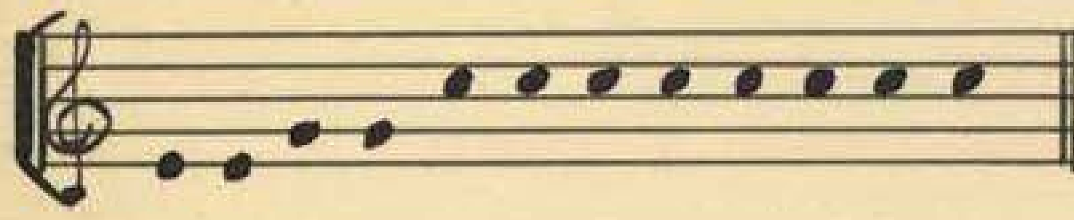

5544 general or 12 circular permutations

5544 general or 12 circular permutations

Total in general permutations: 5544•6 = 32,264

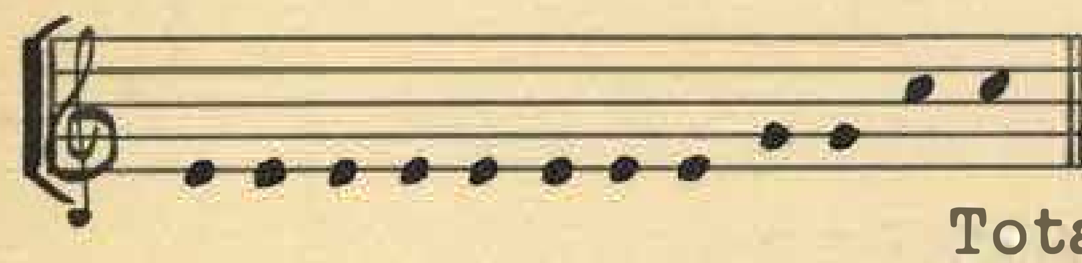
Total in circular permutations: 12•6 = 72

A = 12a; 2a+2b+8c; 2a+8b+2c; 8a+2b+2c



2970 general or 12 circular permutations

2970 general or 12 circular permutations



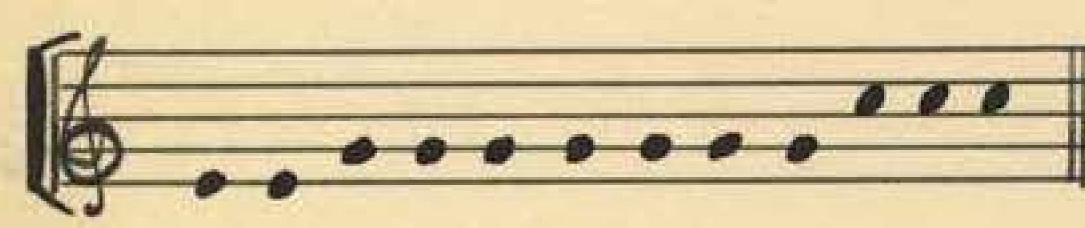
2970 general or 12 circular permutations

Total in general permutations: 2970•3 = 8910

Total in circular permutations: 12•3 = 36

A = 12a; 2a+3b+7c; 2a+7b+3c; 7a+2b+3c;
3a+2b+7c; 3a+7b+2c; 7a+3b+2c

7920 general or 12 circular permutations



7920 general or 12 circular permutations

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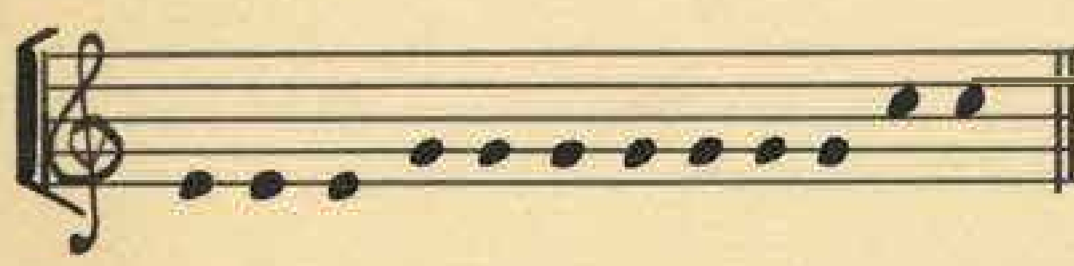


7920 general or 12 circular permutations

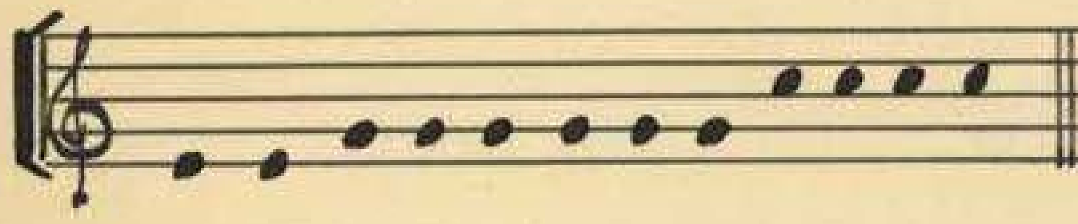
7920 general or 12 circular permutations

Total in general permutations: 7920•6 = 47,520

Total in circular permutations: 12•6 = 72

A = 12a; 2a+4b+6c; 2a+6b+4c; 6a+2b+4c;
4a+2b+6c; 4a+6b+2c; 6a+4b+2c

1386 general or 12 circular permutations



1386 general or 12 circular permutations

1386 general or 12 circular permutations

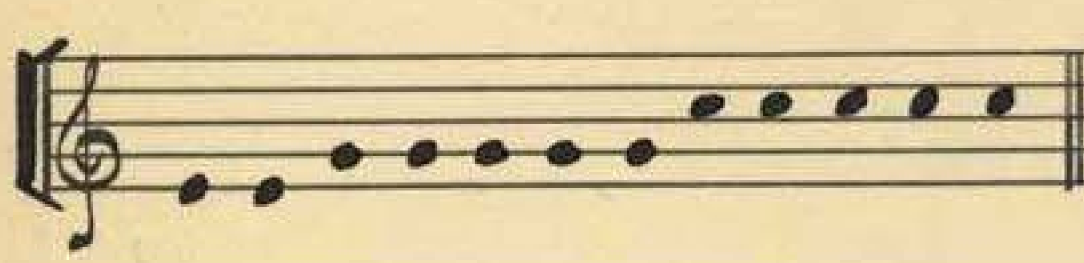
1386 general or 12 circular permutations

1386 general or 12 circular permutations

1386 general or 12 circular permutations

Total in general permutations: 1386·6 = 8316
Total in circular permutations: 12·6 = 72

A = 12a; 2a+5b+5c; 5a+2b+5c; 5a+5b+2c



16,632 general or 12 circular permutations

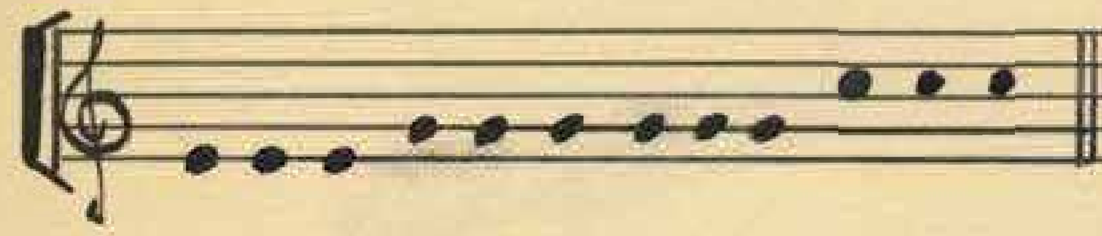
16,632 general or 12 circular permutations

16,632 general or 12 circular permutations

Total in general permutations: 16,632·3 = 49,896
Total in circular permutations: 12·3 = 36

A = 12a; 3a+3b+6c; 3a+6b+3c; 6a+3b+3c

18480 general or 12 circular permutations

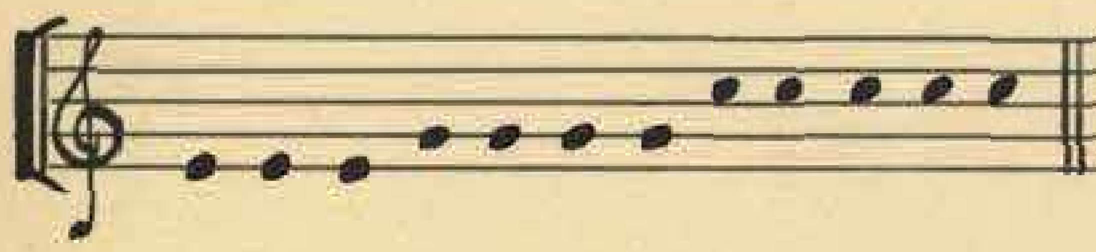


18480 general or 12 circular permutations

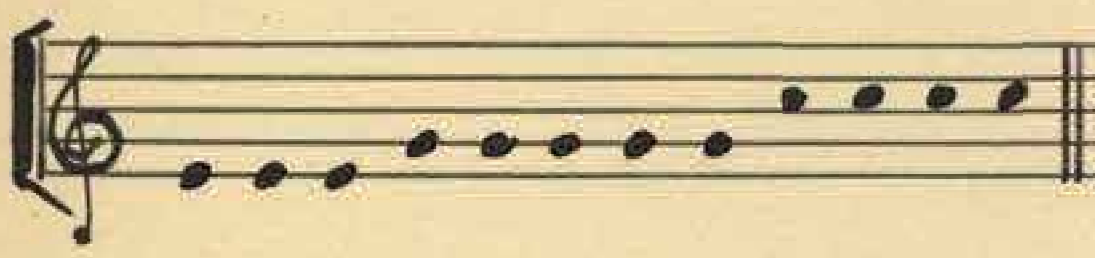
18480 general or 12 circular permutations

Total in general permutations: 18480·3 = 55,440
Total in circular permutations: 12·3 = 36

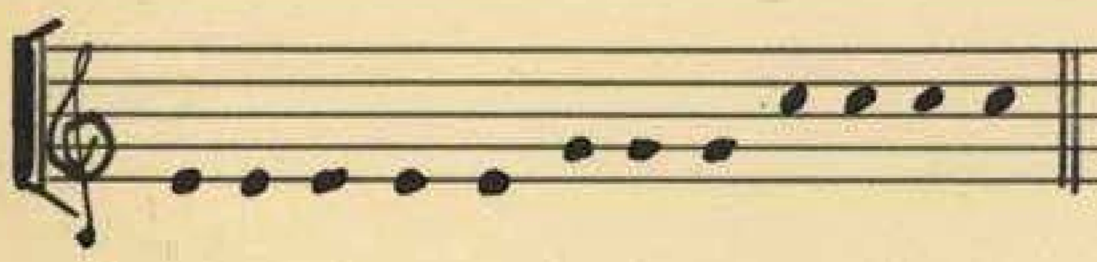
A = 12a; 3a+4b+5c; 3a+5b+4c; 5a+3b+4c;
4a+3b+5c; 4a+5b+3c; 5a+4b+3c



27,720 general or 12 circular permutations

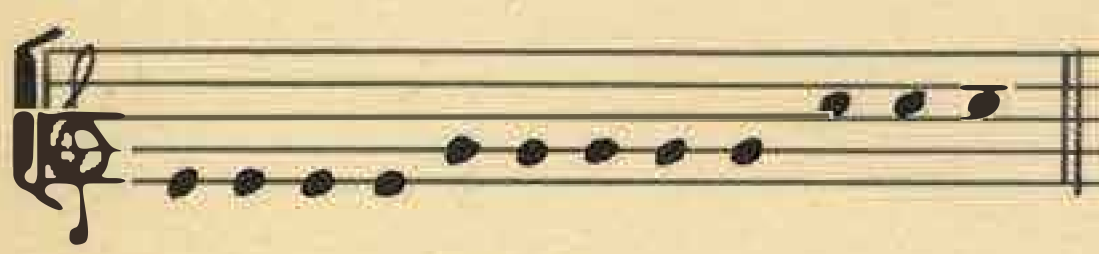


27,720 general or 12 circular permutations

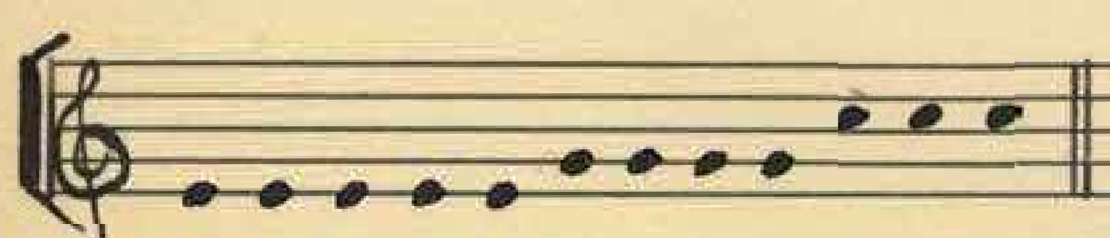


27,720 general or 12 circular permutations

27,720 general or 12 circular permutations



27,720 general or 12 circular permutations



27,720 general or 12 circular permutations

Total in general permutations: 27,720·6 = 166,320
Total in circular permutations: 12·6 = 72

A = 12a; 4a + 4b + 4c.

34,650 general or 12 circular permutations

The entire total for 12 attacks: in general permutations: 443,312
in circular permutations: 660